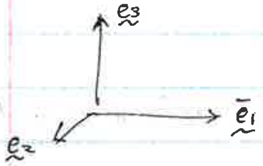
04/01/2024

* Linear & Infinitesimal. → simple

- Tensors { Transformations. ✓
  Stress - strain relations.

Vector : Has magnitude ($\|\vec{V}\|$) and direction ($\vec{V}$) ⇒ $\underset{\sim}{u}$ (notation).



$$\Rightarrow \underset{\sim}{u} = u_1 \underset{\sim}{e_1} + u_2 \underset{\sim}{e_2} + u_3 \underset{\sim}{e_3} = \sum_{i=1}^{3} u_i \underset{\sim}{e_i} = \boxed{u_i \underset{\sim}{e_i}}$$

$(u_1, u_2, u_3 \in \mathbb{R})$

Einstein notation.

$(= u_j \underset{\sim}{e_j})$

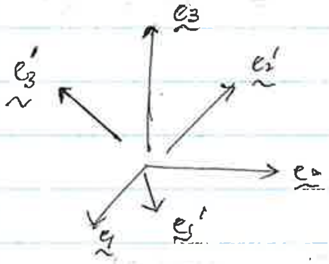
column vector $(\neq \underset{\sim}{u})$

$$\underset{\sim}{u} = u_i \underset{\sim}{e_i} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow \text{Representation of a vector.}$$

(assigns magnitude to the basis $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}$)

↓ depends on basis choice

Coordinate transformation



$$Q_{ij} = (e_i' \cdot e_j') \quad (Q^T = Q^{-1} \Leftrightarrow Q^T Q = I)$$

$$\Rightarrow \boxed{u_i' = Q_{ij} u_j}$$

(j is an index variable).

This is representation vector (not real vector)

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = [Q] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

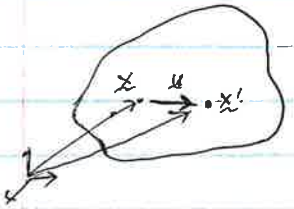
Index notation

$a_{ij} b_j$ → j is dummy (sum over)

$a_i b_j$ → no one is dummy

→ when i or j appears twice, it sums over.

Apply to mechanics.



$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{x}' - \underset{\sim}{x}$$

(very small difference).

<Displacement field>.

$$\partial u_i / \partial x_j \equiv u_{i,j}$$

<Strain field>

Displacement itself can't discriminate translation vs deformation → we need strain field.

$$\Rightarrow \quad u_{i,j} = \frac{\partial u_i}{\partial x_j} \quad \rightarrow \quad \text{strain} \;\; \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ji}$$

$$\text{rotation} \;\; w_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = -\omega_{ji}$$

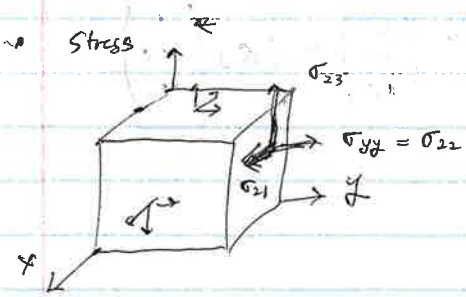
Using $\underset{\sim}{u}_i = Q_{ij}\, \underset{\sim}{u}_j$,

$$\boxed{\varepsilon'_{ij} = Q_{im}\, Q_{jn}\, \varepsilon_{mn}} = Q_{im} \cdot Q_{jn}\, \varepsilon_{mn} = Q\, \varepsilon\, Q^T$$

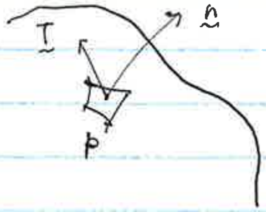
Transpose.

Matrix mul $(Q \cdot \underset{\sim}{\varepsilon})$.

- Stress.



$$\sigma_{ij} = \frac{\text{force in } j^{th} \text{ direction}}{\text{area in } i^{th} \text{ face}}$$ (total 9).



$$T_j = \sigma_{ij}\, \underset{\sim}{n}_i$$

$$\boxed{\sigma'_{ij} = Q_{ip}\, Q_{jq}\, \sigma_{pq}}$$ (same with strain relation).

- Stress — strain relation.

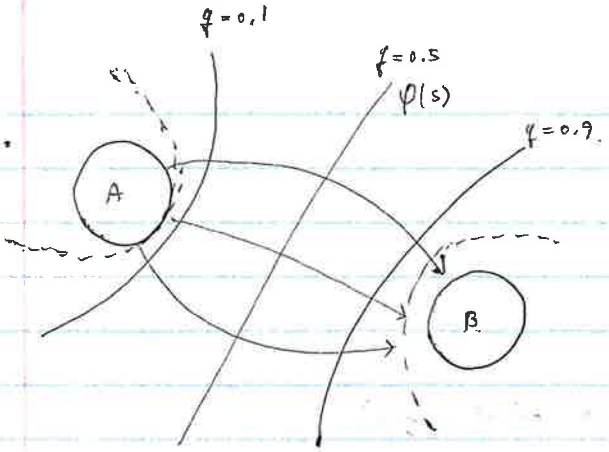
Since the system is linear,

$$\boxed{\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl}.}$$

81 elements. (4! = 24 different numbers).

with coordinate transformation, $C'_{ijkl} = Q_{im} \cdot Q_{jn} \cdot Q_{kp} \cdot Q_{lq} \cdot C_{mnpq}$

04/03/2024.

- 

As long as in probability, there "should be" a region where $P = 0$ .. $(P(A) = P(B) = 0)$ ...

- 

$$\begin{cases} u_i(\underset{\sim}{X}) & : \text{Displacement field.} \\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) & ; \text{ strain field} \\ \sigma_{ij} & ; \text{ stress field} \\ T_j = \sigma_{ij}\, n_i & : \text{ traction force.} \end{cases}$$

- Anisotropic / isotropic elasticity.
  PDE for elasticity → How to solve ? }

- Hooke's law.

$\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{kl}$. (isotropic mat. easy). ⟺ $\varepsilon_{ij} = S_{ijkl}\, \sigma_{kl}$

- Voigt notation.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 \\ \vdots \\ \\ \sigma_6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \\ \varepsilon_6 \end{bmatrix}$$

why is there '2' ?

$$\Rightarrow \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & \cdots & C_{16} \\ \vdots & & \vdots \\ C_{61} & \cdots & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_6 \end{pmatrix}$$

$$\because \sigma_{11} = C_{1111}\,\varepsilon_{11} + \begin{pmatrix} C_{1112}\,\varepsilon_{12} \\ C_{1121}\,\varepsilon_{21} \end{pmatrix} + \cdots +$$

$$\parallel$$

$$C_{1112}\,(2\varepsilon_{12}),$$

$$\begin{pmatrix} C_{11} = C_{1111} \\ C_{16} = C_{1112} \end{pmatrix}$$

$$\Rightarrow \boxed{\sigma_I = C_{IJ}\,\varepsilon_J \quad (I, J = 1, \cdots 6)}$$

$$(\varepsilon_I = S_{IJ}\,\sigma_J)$$

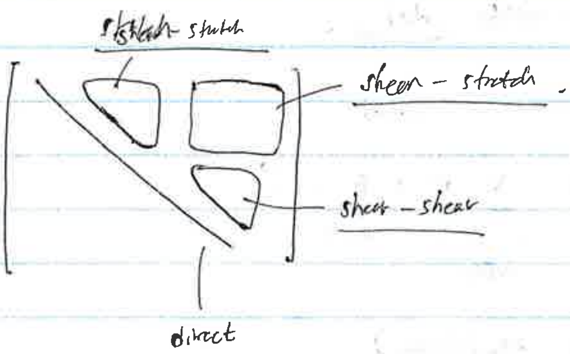
- Elastic stiffness tensor. ✓

$$\begin{array}{cc} & 6 \\ 6 & \begin{bmatrix} \quad C \quad \end{bmatrix} \end{array} \rightarrow \text{symmetric } (\because 2^{nd} \text{ derivative}).$$

σ_{ij} : 9 components → 6 ind comp.

C_{ijkl} : 81 components → 21 ind comp.

$$(6^2 - 6)\cdot \frac{1}{2} + 6 = \underline{21}$$



~~(structure of C)~~

- Isotropic material. ✓

(E, ν, G).

$\langle E = 2(1+\nu) G \rangle$

$S_{11} = S_{22} = S_{33} = 1/E$

$S_{12} = S_{13} = S_{23} = -\nu/E$

$S_{44} = S_{55} = S_{66} = 1/G = 2(1+\nu)/E = 2(S_{11} - S_{12})$

(other $S_{ij} = 0$)

$\sigma_{23} = 2G \cdot \varepsilon_{23}$

(two disappears...)

- $q' = q + C_0$

Two major conditions.

- Compatibility condition. (1)

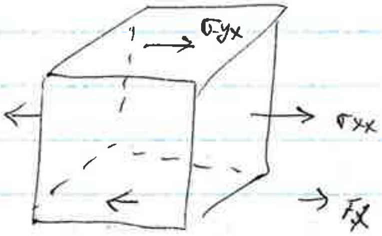
$$\boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0}$$

$\because$ $u_i(\underset{\sim}{x})$: 3 dof and $\varepsilon_{ij}(\underset{\sim}{x})$: 6 dof. (too much)

→ compatibility condition assures it matches dof.

- Equilibrium condition. (2)

$$\boxed{\sigma_{ij,i} + F_j = 0.} \quad \Rightarrow \quad \underbrace{\underset{\sim}{\nabla} \cdot \overset{\text{tensor}}{\underset{\approx}{\sigma}}}_{\text{vector}} + \underbrace{\underset{\sim}{F}}_{\text{vector}} = \vec{0}.$$



(force equilibrium).

- Hooke's law (3)

$$\boxed{\begin{aligned} \sigma_{ij} &= \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \\ \varepsilon_{ij} &= -\frac{\nu}{E} \sigma_{kk} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} \end{aligned}}$$

$(\mu = G)$,

$(\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$.

Note: (1) is satisfied already with $\underline{\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})}$

* General strategies for solution.

① $\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$ → substitute into (3)

$$\Rightarrow \boxed{\mu u_{i,kk} + (\lambda+\mu) u_{k,ki} + F_i = 0}$$

$$\mu \nabla^2 \underset{\sim}{u} \qquad \nabla(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) \qquad \underset{\sim}{F}$$

$$\Rightarrow \mu \nabla^2 \underset{\sim}{u} + \underset{(\lambda+\mu)}{\wedge} \nabla(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{F} = 0. \qquad (3D)$$

approach ② Write compatibility condition in terms of stress (2D)

($\sigma \leftrightarrow \varepsilon$).

∵ In 2D, equil. cond is

$$\left.\begin{aligned} \sigma_{xx,x} + \sigma_{yx,y} + F_x &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + F_y &= 0 \end{aligned}\right\} \boxed{1}$$

compat. cond is

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\,\varepsilon_{xy,xy} = 0. \Big\} \boxed{2}$$

⇒ Trial solution (ansatz).

$$\left\{\begin{aligned} \sigma_{xx} &= \phi_{yy} \\ \sigma_{yy} &= \phi_{xx} \\ \sigma_{xy} &= \phi_{xy} \end{aligned}\right\} \rightarrow \quad \boxed{2} \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \phi(x,y) = 0$$

$$\Rightarrow \nabla^2 \nabla^2 \phi = 0 \Rightarrow \boxed{\nabla^4 \phi = 0}$$

- How to solve elasticity equation. 04/08/2024

: Equations.

$$\begin{cases} \text{compatibility} : & \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \\ \text{equilibrium} : & \sigma_{ij,i} + F_j = 0 \end{cases}$$

: Method:

(1) 3D. $\sigma_{i,j} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$ (only for isotropic)

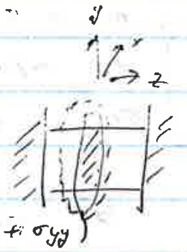
$$\Rightarrow \boxed{\mu u_{i,kk} + (\lambda+\mu) . u_{k,ki} + F_i = 0}$$

$$\Rightarrow \text{Expand}, \begin{cases} \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) . u_x + (\lambda+\mu) \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + F_x = 0 \\ (\text{for } y) \\ (\text{for } z). \end{cases}$$

(2) 2D. $u_x(x,y)$ and $u_y(x,y)$, $\left(u_z(x,y) = 0 , \ \partial/\partial z = 0. \right)$ < Plane strain >

→ no z dependence.

(There is special case, $\boxed{\text{anti plane shear}}$ $\underline{(\text{only non-zero is } u_z(x,y))}$.)



In 2D, ε_{xx} ε_{yy}, $\varepsilon_{xy} \neq 0$, $(\varepsilon_{yz} = \varepsilon_{xz} = \varepsilon_{zz} = 0)$.

σ_{xx} σ_{yy} σ_{zz} $\sigma_{yz} = 0$, $\sigma_{xz} = 0$, $\boxed{\sigma_{zz} \neq 0}$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\Rightarrow \left. \begin{aligned} \varepsilon_{xx} &= \frac{1}{E} \sigma_{xx} - \frac{\nu}{E} (\sigma_{yy} + \sigma_{zz}) \\ \varepsilon_{yy} &= \frac{1}{E} \sigma_{yy} - \frac{\nu}{E} (\sigma_{xx} + \sigma_{zz}) \\ \varepsilon_{xy} &= \frac{1}{2\mu} \sigma_{xy} \end{aligned} \right\}$$

Also, from equilibrium,

$$\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} + F_x = 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + F_y = 0 \end{cases}$$

also, compatibility,

$$\left\{ \varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2 \varepsilon_{xy,xy} = 0. \right.$$

$\updownarrow$ σ_{xx}, σ_{yy}, σ_{zz}. $\sigma_{xz} = 0$ $\sigma_{yz} = 0$ $\sigma_{zz} = 0$ < plane stress >

ε_{xx} ε_{yy} ε_{zz} $\varepsilon_{xz} = 0$ $\varepsilon_{yz} = 0$ $(\varepsilon_{zz} \neq 0)$.

$$\begin{cases} \varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} \\ \varepsilon_{zz} = \frac{1}{2\mu}\sigma_{xy} \end{cases}$$

→ Kolosov constant $k = 3-4\nu$ (plane strain)

$k = \frac{3-\nu}{1+\nu}$ (plane stress)

→ represent with (k)

$\varepsilon_{zz} \neq 0$ makes $\sigma_{zz} \neq 0$ but "$\underline{\sigma_{zz} = 0}$" should be satisfied.

→ compatibility constraint is gone ✓.

For very thin plates, it is okay.

- How to solve?

Ansatz. ϕ where, ϕ is an Airy stress function.

$$\begin{cases} \sigma_{xx} = \phi_{yy} \\ \sigma_{yy} = \phi_{xx} \\ \sigma_{xy} = -\phi_{xy} \end{cases}$$ → Equilibrium condition is automatically satisfied.

$$\Rightarrow \nabla^4 \phi = 0$$

- Examples.

① $\phi(x,y) = \alpha x + \beta y + \gamma \Rightarrow \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$ → trivial solution.

② $\phi(x,y) = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 - Cxy \Rightarrow \sigma_{xx} = B;\ \sigma_{yy} = A,\ \sigma_{xy} = C$

③ 

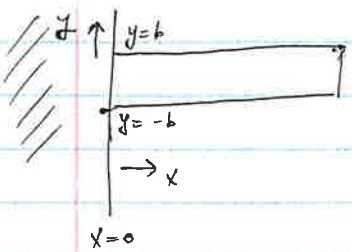
$\sigma_{xx} = \sigma_0 \rightarrow \phi = \frac{1}{2}\sigma_0 y^2$

④ 

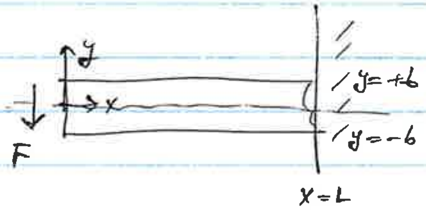
$\phi = -M/I \ \frac{1}{6}y^3$

$\sigma_{xx} = -M/I \cdot y$

- Apply in beam theory.

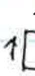


$\left(\sigma_{xx} \Big|_{x=0} = 0 \right)$. or



How to formulate this B.V.P. (4 boundaries)

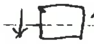
① Top. Bottom. $(y = \pm b)$



$\left.\begin{matrix} \sigma_{yy} = 0 \\ \sigma_{xy} = 0 \end{matrix}\right\}$

⇒ strong B.C. (for every point)

② Left. $(x=0)$



$\int_{-b}^{+b} \sigma_{xy} \, dy = F$,

⇒ weak B.C. (integral).

$\int_{-b}^{b} \sigma_{xx} \, y \, dy = 0$ (moment)

$\int_{-b}^{b} \sigma_{xx} \, dy = 0$

③ Right $(x=L)$

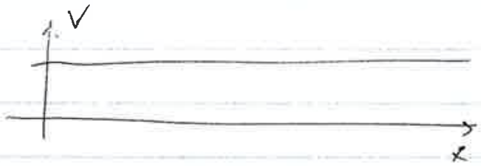
"Automatically satisfied" (only weak ones)

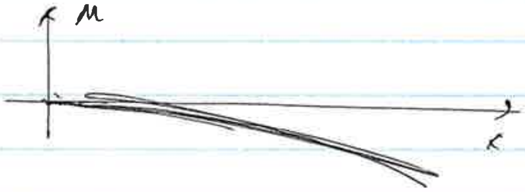
(∵ Stress is propagated to adjacent material elements)

force, moment balance (equilibrium).

Actual B.C $(x=L)$ is.

$\left(u_x = 0 \right)$
$\left(u_y = 0 \right)$ → weak B.C may not be satisfied





Guess : $\phi = C_1 xy^3 \Rightarrow \left.\begin{matrix} \sigma_{xx} = 6C_1 xy \\ \sigma_{yy} = 0 \\ \sigma_{xy} = -3C_1 y^2 \end{matrix}\right\}$

$\sigma_{xy} = -3C_1 b^2$ $(y = \pm b)$ → Add $+3C_1 b^2$ to σ_{xy}

$\Rightarrow \boxed{\phi = C_1 xy^3 - 3C_1 b^2 xy}$ (fixed form).

$\left.\begin{matrix} \sigma_{xx} = 6C_1 xy \\ \sigma_{yy} = 0 \\ \sigma_{xy} = -3C_1 y^2 + 3C_1 b^2 \end{matrix}\right\}$ → Satisfied! → $x=0$, $\int_{-b}^{b} \sigma_{xy} \, dy = F \rightarrow C_1 = F/(4b^3)$

$$\{ p_s , (1-p_s)\cdot p_s , (1-p_s)^2 p_s , \cdots \}$$

$$\textcircled{t_0} = \begin{cases} \langle t_s \rangle & \text{w.p. } p_s \quad \{\langle t_s\rangle , \langle t_F\rangle + \langle t_s \rangle , 2\langle t_F\rangle + \langle t_s\rangle \\ \langle t_F \rangle & \text{w.p. } 1-p_s \left[p_s(1-p_s)\cdot\langle t_F\rangle \{ 1 + 2(1-p_s) + 3(1-p_s)^2 + \cdots \} \right. \end{cases}$$

04/10/2024

$\langle t \rangle = ?$ we know : $\langle t_F \rangle$ (by experiment).

$$(1-p_s)\langle t_F \rangle + \langle t_s \rangle \cdot p_s =$$

$$\underline{\frac{1-p_s}{p_s}\cdot\langle t_F\rangle + \langle t_s\rangle} \; ?$$

$\phi(x,y)$

$\downarrow$ $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$

$$\rightarrow \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} = \sim$$

$$\Rightarrow u_x = \int \varepsilon_{xx} dx = \frac{3F}{4Eb^3} x^2 y + \boxed{f(y)}$$

$$u_y = \int \varepsilon_{yy} dy = -\frac{3F}{4Eb^3} x y^2 + \boxed{g(x)}$$

arbitrary functions

$$\varepsilon_{xy} = \frac{1}{2}\left(u_{x,y} + u_{y,x}\right) = \frac{1}{2}\left(\frac{3F}{4Eb^3}x^2 - \frac{3F}{4Eb^2}y^2\right) + \frac{1}{2}\left(f'(y) + g'(x)\right) \quad \text{—①}$$

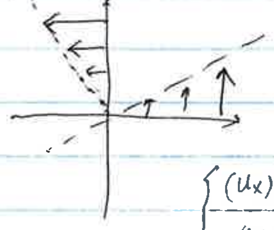
$$\therefore \varepsilon_{xy} = \frac{3F(1+\nu)}{4Eb^3}(b^2-y^2) = ①$$

$$\Rightarrow \frac{3F}{4Eb^3}x^2 + g'(x) = \frac{3F(1+\nu)}{4Eb^3}(b^2-y^2) + \frac{3F\nu}{4Eb^3}y^2 - f'(y) = \boxed{\text{Const}} = c$$

$\because$ Must be for any x, y

$(u_x = -cy,\ u_y = 0)$

Note : $u_x = \cdots - cy$
$u_y = \cdots + cx$



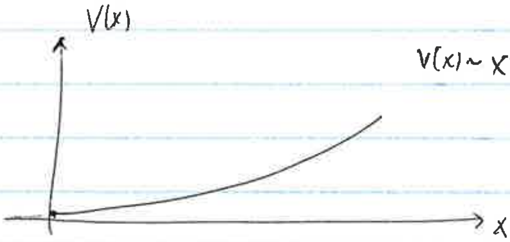
$\Rightarrow$ This is "rotation"

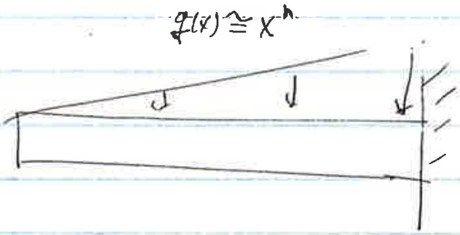
$$\begin{cases} (u_x)^2 + (u_y)^2 = c^2(x^2+y^2) \\ u_y/u_x = -x/y \end{cases}$$

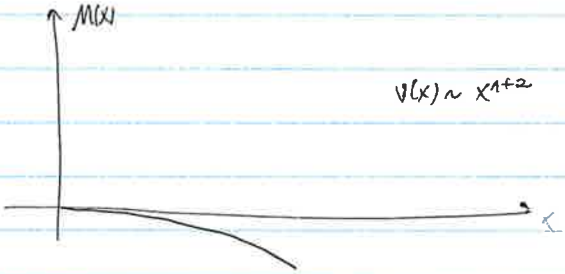
* we can add correction term ($\because$ currently $u_i \neq 0$ at $x=L$)

→ The correction term decays as it goes away from $x=0$.

$\therefore$ F, M at $x=0$ is same effect at ($x=L$) Same as principle







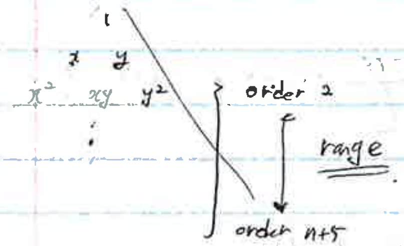
$$\sigma_{xx} \sim x^{n+2} y$$

$$\hookrightarrow \underline{\phi \sim x^{n+2} y^3}$$

(max order n+5)

(p_1, (p_2) ...

we guess $\phi(x,y) = C_1x^2 + C_2xy + C_3y^2 + C_4x^3$



($\nabla^4\phi = 0$ must hold)

Comp

compu

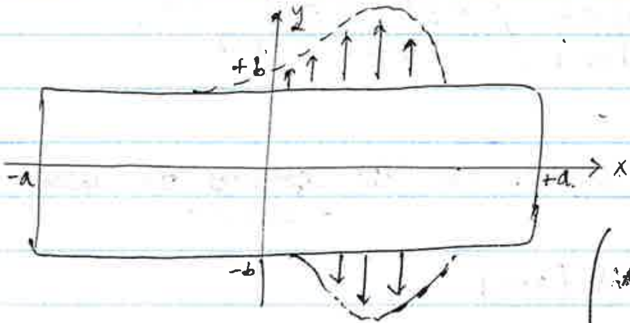
comp

compat

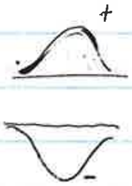
- Fourier solution. (2D elasticity problem) 04/15/2014.

stress function $\phi(x,y)$ → $\sigma_{xx} = \phi_{,yy}$, $\sigma_{yy} = \phi_{,xx}$, $\sigma_{xy} = -\phi_{,xy}$ } Equilibrium satisfied; compatibility : $\nabla^4\phi = 0$

How to solve ? → Polynomials vs. Fourier method. (nicer)
(∵ we can do this since the system is Linear).



$\sigma_{yy}(x, y=\pm b) = \pm t_{y\pm}(x)$
$\sigma_{xy}(x, y=\pm b) = \pm t_{x\pm}(x)$

(Note :


at bottom is (−) σ_{yy} (compress)
at bottom is (+). σ_{yy} (stretch).)

Ansatz: $\phi(x,y) = e^{\alpha x} e^{\beta y}$ $(\alpha, \beta \in \mathbb{C})$ 복소수. ⇒ $\cos x + j\sin x$ (can express all numbers).

$\nabla^2\phi = (\alpha^2+\beta^2)\,\phi(x,y)$

$\nabla^4\phi = [\alpha^2(\alpha^2+\beta^2) + \beta^2(\alpha^2+\beta^2)]\,\phi(x,y) = (\alpha^2+\beta^2)^2\,\phi(x,y) = 0$

⇒ $\alpha^2+\beta^2=0$ ⇒ $\alpha=\lambda$, $\beta=j\lambda$ (where $\lambda \in \mathbb{R}$) or others...

~~Smooth~~ ∴ $\phi(x,y) = e^{j\lambda x}e^{\lambda y},\ e^{j\lambda x}e^{-\lambda y},\ e^{j\lambda x}\,y\,e^{\lambda y},\ e^{j\lambda x}\,y\,e^{-\lambda y}$ (prepare for $a \gg b$)

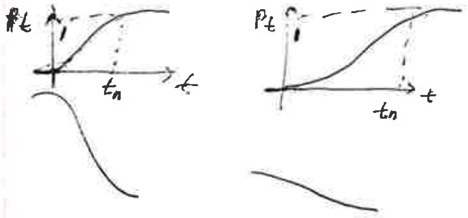
actually there are 4 more solutions ... (interchange x and y).

$$\Rightarrow \phi(x,y) = e^{j\lambda x}\{(c_1+c_2y)e^{\lambda y} + (c_3+c_4y)e^{-\lambda y}\}.$$ (4 DOF) { Top/Bottom ✓, Left/Right }

$$= e^{j\lambda x}\{A\cosh\lambda y + B y\cosh\lambda y + C\sinh\lambda y + D y\sinh\lambda y\}$$

$\phi(x,y) = \cos\lambda x\,\{A'\cosh\lambda y + D' y\sinh\lambda y\}$ (even x even y)
$+ \cos\lambda x\,\{B'\sinh\lambda y + C' y\cosh\lambda y\}$ (even x odd y)
$+ \sin\lambda x\,\{\quad " \quad\}$ (odd x even y)
$+ \sin\lambda x\,\{\quad " \quad\}$ (odd x odd y)

} Use one of these depending on symmetry.

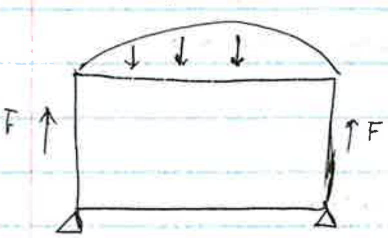


$t_n << t_n$

$P_t = \{ P_\tau \mid \tau \leq t \}$

rate = $1/t_n$ (t_n when $P_t \to 1$). ?

- Example.

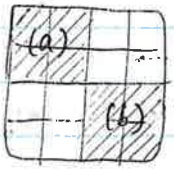


$P_y(x) = p_0 \cos\left(\frac{\pi x}{2a}\right)$

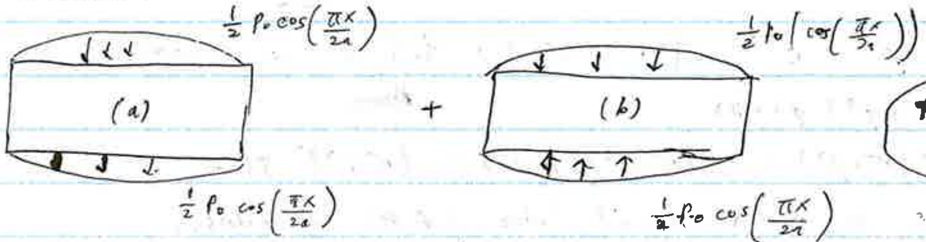
$\boxed{\text{S-t Venant}}$!

$\sigma_{yy}\Big|_{y=b} = -p_0 \cos\left(\frac{\pi x}{2a}\right)$ / $\sigma_{yy}\Big|_{y=-b} = 0$ / $\sigma_{xy}\Big|_{y=b} = \sigma_{xy}\Big|_{y=-b} = 0$.

With $\lambda = \frac{\pi}{2a}$, find C_1, C_2, C_3, C_4 $\longrightarrow$ Use symmetry to solve.



[4×4] ⇒ double [2×2]



Two symmetric problems

ϕ < even x odd y > ϕ < even x even y >

σ_{yy} : even x odd y
σ_{xx} : even x odd y
σ_{xy} : odd x even y

(a) $\phi = \cos\lambda x \{ B y \cosh\lambda y + C \sinh\lambda y \}$ ($\lambda = \pi/(2a)$)

B.C.: $\sigma_{xy} = 0$
$\sigma_{yy}\Big|_{y=b} = -\frac{1}{2} p_0 \cos\lambda x$ $\Rightarrow$ Find (B, C)

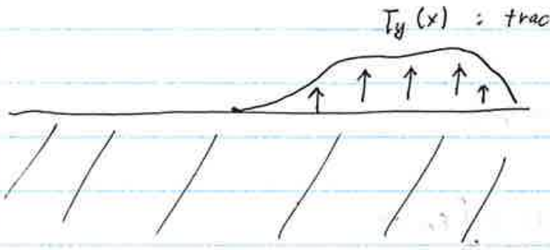
(b) $\phi = \cos\lambda x \{ D y \sinh\lambda y + A \cosh\lambda y \}$ ($\lambda = \pi/(2a)$)

B.C. $\sigma_{xy} = 0$
$\sigma_{yy} = -\frac{1}{2} p_0 \cos\lambda x$ $\Rightarrow$ Find (A, D)

$\rightarrow$ Merge A, B, C, D $\rightarrow$ $\phi(x,y)$

4/17/2024

- Elastic space.



$\textcircled{A} x = \lambda x$

Find displacement x due to force at x'

$$u(x, x') = \underbrace{T(x, x')}_{\text{operator}} \cdot T_y(x')$$

Green's function

$$u(x) = \int_\Omega u(x,x')\,dx' = \underbrace{\int_\Omega G_s(x-x')\,T_y(x')\,dx'}_{\text{Convolution}}$$

$$\Rightarrow \boxed{u_i(x) = \int_\Omega G_{s\,ij}(x-x')\,T_j(x').\,dx'}$$ (assumption : under plane strain)

Example1) when $T_y(x) = e^{jkx}$

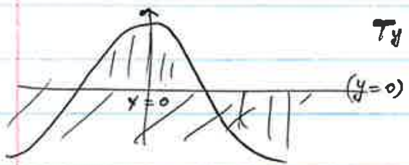
$$\Rightarrow u_y(x) = \int_{-\infty}^{\infty} G_s(x-x')\,e^{jkx'}dx' = e^{jkx}\underbrace{\int_{-\infty}^{+\infty} G_s(\tau)\,e^{-jk\tau}\cdot d\tau}$$

< Fourier transform of $G_s(x)$ >

《 Recall that $e^{jkx} \xrightarrow{T} F\{T\}.e^{jkx}$ since e^{jkx} is eigen function 》

$$\Rightarrow u_y(x) = \hat{G_s}(k)\,T_y(x)$$

Example 2)



$$T_y(x) = T_0\cos(\lambda x) = T_0\left\{\frac{e^{j\lambda x}+e^{-j\lambda x}}{2}\right\}$$

Care $-y$ since we are "half space"

$$\phi(x,y) = \cos\lambda x\,\{A+By\}\,e^{\lambda y}$$

$\left(\because \lim_{y\to-\infty} e^{-\lambda y} \gg \infty\right)$

$$\left.\begin{aligned}\sigma_{yy}(x, y=0) &= T_0\cos\lambda x.\\ \sigma_{xy} &= 0 \quad (x, y=0)\end{aligned}\right\}$$

$$\left.\begin{aligned}\sigma_{yy} &= -\lambda^2\cos\lambda x\,(A+By).e^{\lambda y}\\ \sigma_{xy} &= \lambda\sin\lambda x\,(A\lambda+B+B\lambda y).e^{\lambda y}\end{aligned}\right\} \rightarrow B=-A\lambda$$

$$\left(A = -T_0/\lambda^2 \quad B = T_0/\lambda\right)$$

$$\sigma_{xx} = \cos\lambda x\,(A\lambda^2+2B\lambda+B\lambda^2 y).e^{\lambda y}$$

→ Continued...

Thus, $\begin{cases} \sigma_{xx} = T_0 \cos\lambda x \;(1+\lambda y)\; e^{\lambda y} \\ \sigma_{yy} = T_0 \cos\lambda x \;(1-\lambda y)\; e^{\lambda y} \\ \sigma_{xy} = T_0 \cdot \lambda \sin\lambda x \;\; y\, e^{\lambda y} \end{cases}$

Using plane strain assumptions, (+ isotropic)

$$\left.\begin{aligned} \varepsilon_{xx} &= 1-\sim \quad \sim \\ \varepsilon_{xy} &= \sim \\ \varepsilon_{yy} &= \sim \end{aligned}\right\} \text{Notes..} \Rightarrow \begin{aligned} u_x &= \int \varepsilon_{xx}\, dx \\ u_y &= \int \varepsilon_{yy}\, dy \\ \varepsilon_{xy} &= \tfrac{1}{2}(u_{x,y} + u_{y,x}) = \sim \end{aligned} \Rightarrow C = D = \text{Const.}$$

$$u_x(x, y=0) = \frac{T_0}{\lambda E} \cdot \sin\lambda x \;(1-\nu-2\nu^2)$$

$$u_y(x, y=0) = \frac{T_0}{\lambda E} \cdot \cos\lambda x \;(2-2\nu^2) = T_y(x) \cdot \boxed{\frac{2-2\nu^2}{\lambda E}}$$

$$\Rightarrow G_{S_{yy}}(k) = \frac{2(1-\nu^2)}{kE} = \frac{2(1-\nu^2)}{k\, 2\mu(1+\nu)} = \frac{1-\nu}{|k| \mu}$$

$$\cos kx \longrightarrow \sin kx \; : \; \underline{x' = x - \frac{\pi}{2k}}$$

$$G_{S_{yy}}(x) \Rightarrow F^{-1}\left[\frac{1}{|k|}\right] \frac{1-\nu}{\mu} = -\frac{1}{\pi}(\ln x) \frac{1-\nu}{\mu} = -\frac{1-\nu}{\mu\pi} \ln x$$

~

$$\kappa = 3-4\nu \;(\text{Kosonov's constant}) \Rightarrow \underline{G_{S,yy}(x) = -\frac{\kappa+1}{4\mu\pi} \ln x}$$

dt same.

[Note: Since we have loading in y, G_S is corresponds to $\underline{u_y(x)}$

$\underset{\sim\sim\sim}{T_y(x) \rightarrow u_y(x)}$]

$$u_x = \frac{T_0 \ \sin ky}{2\mu k} \cdot (1 - e\nu)$$

$$T_y = e^{jkx} = \cos kx + j \sin kx$$

$$u_x = A(\sin kx - j \cos kx).$$

$$u_x = -jA\left(\cos kx - \frac{1}{j}\sin kx\right) \equiv -jA\, e^{jkx}$$

$$\Rightarrow \underset{\sim\sim\sim}{\mathcal{F}^{-1}\{\hat{G}_{s\,xy}(k)\}} = \mathcal{F}^{-1}\left\{-\frac{(1-2\nu)}{2\mu}\left(i/k\right)\right\} = +\frac{(k-1)}{4\mu}\cdot\frac{1}{2}\ \mathrm{sgn}(x).$$

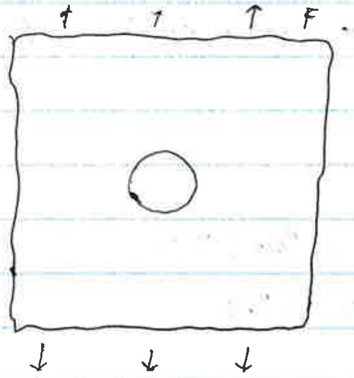
Summary

		$\mathbb{F}$	$\mathbb{R}$
y-load	$G_{s\ yy}$	$\frac{k+1}{4\mu}\cdot\frac{1}{\lvert k\rvert}$	$-\frac{k+1}{4\pi\mu}\ln x$
	$G_{s\ xy}$	$-\frac{(k-1)}{4\mu}\left(\frac{i}{k}\right)$	$+\frac{(k-1)}{8\mu}\ \mathrm{sgn}(x).$
x-load	$G_{s\ xx}$	$\frac{k+1}{4\mu}\cdot\frac{1}{\lvert k\rvert}$	$-\frac{k+1}{4\pi\mu}\ln x$
	$G_{s\ xy}$	$\frac{k-1}{4\mu}\left(i/k\right)$	$-\frac{(k-1)}{8\mu}\cdot\mathrm{sgn}(x).$

$$T = T_y\,\vec{e_y} + T_x\,\vec{e_x}$$

$$= \int_{-\infty}^{\infty} -\frac{k+1}{4\pi\mu}\,\underline{\ln(x-x')\,T_x(x')\,dx'} + \int_{-\infty}^{\infty}\frac{k-1}{8\mu}\ \mathrm{sgn}(x-x')\cdot T_y(x')\,dx'.$$

- Polar coordinates. (r, θ, ϕ) 04/22/2024



$r = \sqrt{x^2+y^2}$ $\phi(x,y) \longrightarrow \phi(r,\theta).$

$\theta = \tan^{-1}(y/x)$

$$\left.\begin{aligned} \sigma_{yx} &= \phi_{,yy} \\ \sigma_{yy} &= \phi_{,xx} \\ \sigma_{xy} &= \phi_{,xy} \end{aligned}\right\} \quad \left\{\begin{aligned} \sigma_{rr} &= \frac{1}{r}\partial\phi/\partial r + \frac{1}{r^2}\partial^2\phi/\partial\theta^2 \\ \sigma_{\theta\theta} &= \partial^2\phi/\partial r^2 \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\partial\phi/\partial\theta\right) \end{aligned}\right\}$$

- Derivation.

Idea ①: $(r,\theta) \rightarrow \phi(r,\theta) = \phi(x,y) \rightarrow (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \xrightarrow{\text{Coordinate trans}} \rightarrow \rightarrow \underline{\underline{(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})}}$

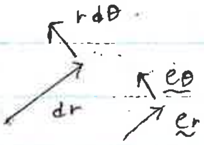
Idea ②: Tensor calculus.

$$\underset{\sim}{\nabla} = \underset{\sim}{e_x}\frac{\partial}{\partial x} + \underset{\sim}{e_y}\frac{\partial}{\partial y}. \quad \text{(grad operator)}$$

$$= \underset{\sim}{e_{x'}}\frac{\partial}{\partial x'} + \underset{\sim}{e_{y'}}\frac{\partial}{\partial y'} \quad \text{(}\underline{\text{Independent}}\text{ of coordinate system)}$$

$\nabla^2 = \nabla \cdot \nabla$ (Laplacian — also $\underline{\text{independent}}$ of coordinate system)

In polar coordinates.



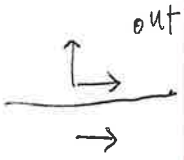
$$\Rightarrow \underset{\sim}{\nabla} = \underset{\sim}{e_r}\,\partial/\partial r + \underset{\sim}{e_\theta}\cdot\frac{1}{r}\,\partial/\partial\theta.$$

also, $\Rightarrow \left(\dfrac{\partial \underset{\sim}{e_r}}{\partial\theta} = \underset{\sim}{e_\theta} \text{ and } \dfrac{\partial \underset{\sim}{e_\theta}}{\partial\theta} = -\underset{\sim}{e_r}\right)$ ☆

↳ $\underset{\sim}{e_r}$ and $\underset{\sim}{e_\theta}$ $\underline{\underline{\text{rotates}}}$ around.

$$\Rightarrow \nabla^2 = \nabla\cdot\nabla = \left(\underset{\sim}{e_r}\,\partial/\partial r + \underset{\sim}{e_\theta}\,\frac{1}{r}\,\partial/\partial\theta\right)\cdot\left(\underset{\sim}{e_r}\,\partial/\partial r + \underset{\sim}{e_\theta}\,\partial/\partial\theta\right)$$

$$= \partial^2/\partial r^2 + \underbrace{\frac{1}{r}\,\partial/\partial r}_{\text{extra term.}} + \frac{1}{r^2}\,\partial^2/\partial\theta^2$$



0.

derivation continued...

we cannot use $\underset{\approx}{\sigma} \neq \underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla} \phi$ (no minus in σ_{xy}) $= (-)\phi$

$\Rightarrow \underset{\sim}{\nabla}^{a} = \underset{\sim}{\nabla} \times \underset{\sim}{e_z} = -\underset{\sim}{e_y}\, \partial/\partial x + \underset{\sim}{e_x}\, \partial/\partial y$

Thus, $\boxed{\underset{\approx}{\sigma} = \underset{\sim}{\nabla}^{a} \otimes \underset{\sim}{\nabla}^{a} \phi}$

In polar coordinate, $\nabla^{a} = \left(\underset{\sim}{e_r} \cdot \partial/\partial r + \underset{\sim}{e_\theta} \cdot \frac{1}{r}\, \partial/\partial\theta \right) \times \underset{\sim}{e_z}$

$= -\underset{\sim}{e_\theta} \cdot \partial/\partial r + \underset{\sim}{e_r} \cdot \frac{1}{r}\, \partial/\partial\theta$

Now, we solve

(1) $\nabla^4 \phi = 0$

$\Rightarrow \left(\partial^2/\partial r^2 + \frac{1}{r}\, \partial/\partial r + \frac{1}{r^2}\, \partial^2/\partial\theta^2 \right)^2 \phi = 0$

(2) Disp. - Strain.

$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$

$\underset{\approx}{\varepsilon} = \frac{1}{2} \left\{ \underset{\sim}{\nabla} \otimes \underset{\sim}{u} + (\underset{\sim}{\nabla} \otimes \underset{\sim}{u})^T \right\}$

$$\Rightarrow \begin{cases} \varepsilon_{rr} = \partial u_r/\partial r \\ \varepsilon_{\theta\theta} = \frac{1}{r}\, \partial u_\theta/\partial\theta + \frac{1}{r} u_r \\ \varepsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial\theta} - \frac{1}{r} u_\theta + \partial u_\theta/\partial r \right) \end{cases}$$

(3) Gen. Hooke's law.

$\underset{\approx}{\sigma} = \lambda\, \mathrm{Tr}[\underset{\approx}{\varepsilon}]\, \underset{\approx}{I} + 2\mu\, \underset{\approx}{\varepsilon}$

(4) Trac. force vs stress.

$\underset{\sim}{T} = \underset{\sim}{n} \cdot \underset{\approx}{\sigma} \Leftrightarrow T_j = \sigma_{ij} n_i$

(5) Equil. condition

$\sigma_{ij,i} + F_j = 0$:

$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \underset{\sim}{F} = 0$.

$\rightarrow$ $\boxed{\text{Already satisfied}}$ since we started from $\phi_{,yy} = \sigma_{xx}$, $\phi_{,xx} = \sigma_{yy}$ ...

($\because$ Define σ from ϕ (stress function))

- Solve.

$\nabla^4 \phi (r, \theta) = 0$. expect $\phi(\theta) = \phi(\theta + 2\pi)$

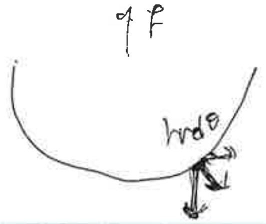
$\phi(r,\theta) = R(r) \cdot \Theta(\theta) = f(r) \cdot e^{jn\theta}$ $(n \in \mathbb{Z})$. $(n = 0, 1, 2, \cdots)$ $\left(\begin{array}{l} \partial/\partial\theta\, \phi = jn\phi \\ \partial^2/\partial\theta^2\, \phi = -n^2\phi \end{array} \right)$.

$\Rightarrow \left(\partial^2/\partial r^2 + \frac{1}{r}\, \partial/\partial r - n^2/r^2 \right) \left(\partial^2/\partial r^2 + \frac{1}{r}\, \partial/\partial r - n^2/r^2 \right) f(r) = 0$.

$\Rightarrow$ set Ansatz $f(r) \sim r^m \Rightarrow (m^2 - n^2)((m-2)^2 - n^2)\, \underline{r^{m-4}} = 0$.

$m^2 - n^2 = 0$ or $(m-2)^2 - n^2 = 0$

$\Rightarrow$ $\underline{m = \pm n \text{ or } m = 2 \pm n}$ //



$1/r$

Using $m = \pm n$, $m = 2 \pm n$ for $f(r) \sim r^m$.

$\Rightarrow \phi(r,\theta) = (A_n r^{2+n} + B_n r^{2-n} + C_n r^n + D_n r^{-n}) \underset{\sim}{e^{jn\theta}}$ (for each n)

→ we need 4 boundary conditions.

Problem : when $n=0$, $m = 0, 0, 2, 2$ (only 2)

$n=1$, $m = 1, -1, 1, 3$ (two overlaps)

↳ No more 4 B.C.S.

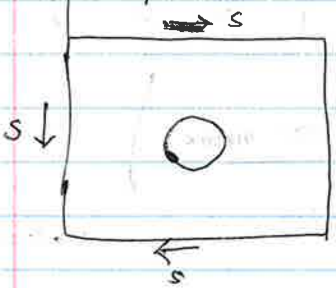
→ Solution : Michell solution's table.

$$\begin{cases} f_0(r) = A_0 r^2 + B_0 r^2 \ln r + C_0 \ln r + D_0 \\ f_1(r) = A_1 r^1 + B_1 r^{-1} + C_1 r^1 \ln r + D_1 r^3 \end{cases}$$

$(r^m \ln r, \ r^m)$ both are solutions.

(E.g. $\partial/\partial n \cdot r^n = \partial/\partial n \cdot e^{n \ln r}$
$= \ln r \cdot r^n$)

Real problems (examples).



$$\begin{cases} \sigma_{xx} = \sigma_{yy} = 0 & (r \to \infty) \\ \sigma_{xy} = S & (r \to \infty) \end{cases} \quad ①$$

$$\begin{cases} \sigma_{r\theta} = 0 & (r = a) \\ \sigma_{rr} = 0 & (r = a) \end{cases} \quad ②$$

start from $\phi = \phi^{(0)} + \phi^{(1)}$

$\phi^{(0)} = -Sxy \Rightarrow \sigma_{xy}^{(0)} = S \Rightarrow \phi^{(0)} = -S r^2 \cos\theta \sin\theta$

Then,
$$\left.\begin{aligned} \sigma_{rr}^{(0)} &= S \sin 2\theta \\ \sigma_{\theta\theta}^{(0)} &= -S \sin 2\theta \\ \sigma_{r\theta}^{(0)} &= S \cos 2\theta \end{aligned}\right\} \text{ satisfies ① but not ②}$$

$$\phi^{(1)} = (\underbrace{A_2 r^4 + B_2 r^2}_{\text{not}} + C_2 + D_2 r^{-2}) \sin(2\theta)$$

not ~~necessary (r⁴ blows up, r² is in~~ σxy)

$$\underbrace{\left.\begin{aligned} \Rightarrow \sigma_{rr}^{(1)} &= \left(S - \frac{4A}{r^2} - \frac{6B}{r^4} \right) \cdot \sin 2\theta \\ \sigma_{r\theta}^{(1)} &= \left(S + \frac{2A}{r^2} + \frac{6B}{r^4} \right) \cos 2\theta \\ \sigma_{\theta\theta}^{(1)} &= \left(-S + \frac{6B}{r^4} \right) \sin 2\theta \end{aligned}\right\}}_{\text{Effect of the hole}}$$

$r = a$ → plug in → solve $\boxed{A, B}$

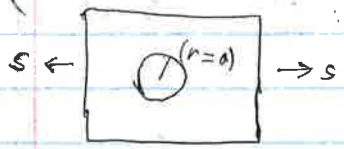
$(A = Sa^2, \ B = -\frac{1}{2} Sa^2)$

✓ $\begin{cases} \sigma_{xy} = \sigma_{yy} = 0 & (r \to \infty) \\ \sigma_{xx} = S & (r \to \infty) \end{cases}$

✓ $\begin{cases} \sigma_{rr} = 0 & (r = a) \\ \sigma_{r\theta} = 0 & (r = a) \end{cases}$

Example 1 – extra Guess : $\phi^{(0)} = \frac{1}{2} S y^2 \longrightarrow \sigma_{xx} = \phi_{,yy} = S.$

Add B.C. $\phi^{(1)} = ?$



Note $\phi^{(0)} = \frac{1}{2} S y^2 = \frac{1}{2} S \cdot (r \sin\theta)^2 = \frac{1}{4} S r^2 (1 - \cos 2\theta)$
$\downarrow$
Mitchell 0th and 2nd

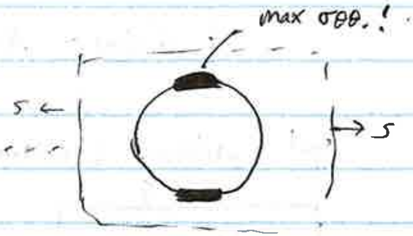
$\Rightarrow \phi^{(1)} = A \ln r + B\theta + C \cdot \cos 2\theta + D \sin 2\theta / r^2 = \phi^{(1)}(r, \theta)$

Thus, $A = -Sa^2/2$, $B = 0$, $C = Sa^2/2$, $D = -Sa^4/4$

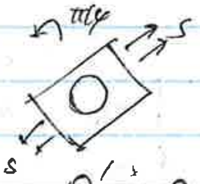
$\therefore \phi(r,\theta) = \phi^{(0)}(r,\theta) + \phi^{(1)}(r,\theta)$

$\Rightarrow \sigma_{\theta\theta}(r = a) = S(1 - 2\cos 2\theta)$

$\max(\sigma_{\theta\theta}) = 3S$ (at $\theta = \pi/2$, $3\pi/2$, ...)



→ (It makes sense at



with rotation matrix)

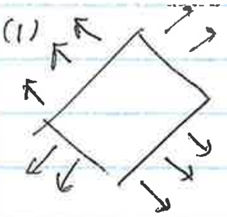
~~Gauss~~

Example 2

$\theta' = \theta + \pi/4$ (anti-clockwise)

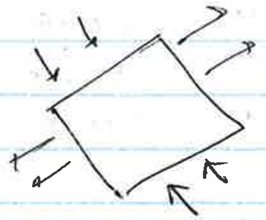
$\theta'' = \theta - \pi/4$ (clock-wise)

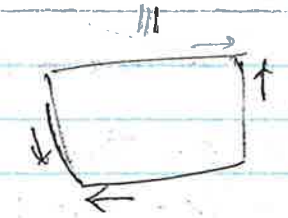
(1)



$\sigma_{r\theta} = 0$
$\sigma_{rr} = S(1 - a^2/r^2)$
$\sigma_{\theta\theta} = S(1 + a^2/r^2)$

(2)



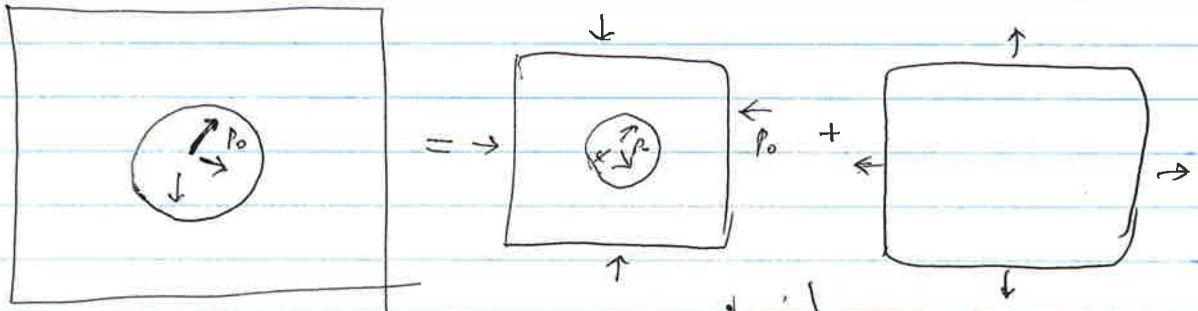


$\sigma_{rr} =$
$\sigma_{r\theta} =$ } Same as shear
$\sigma_{\theta\theta} =$

Why ?

Qual exam!

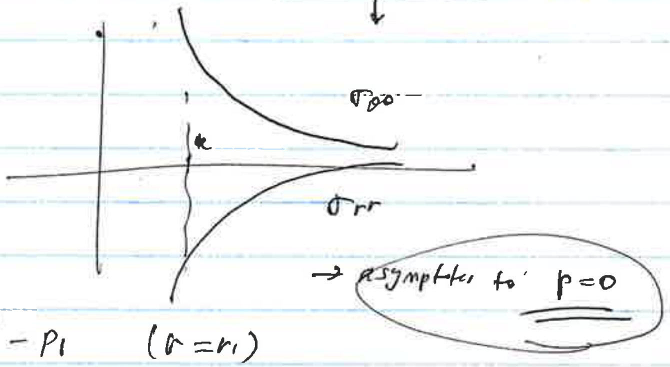
①



$\sigma_{rr} = -p_0 + p_0(1 - a^2/r^2)$

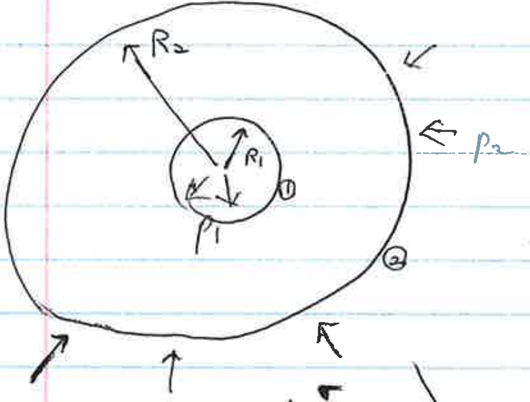
$\sigma_{\theta\theta} = -p_0 + p_0(1 + a^2/r^2)$

$\sigma_{r\theta} = \sim$



→ asymptotes to $p=0$

② Thick walled pressure.

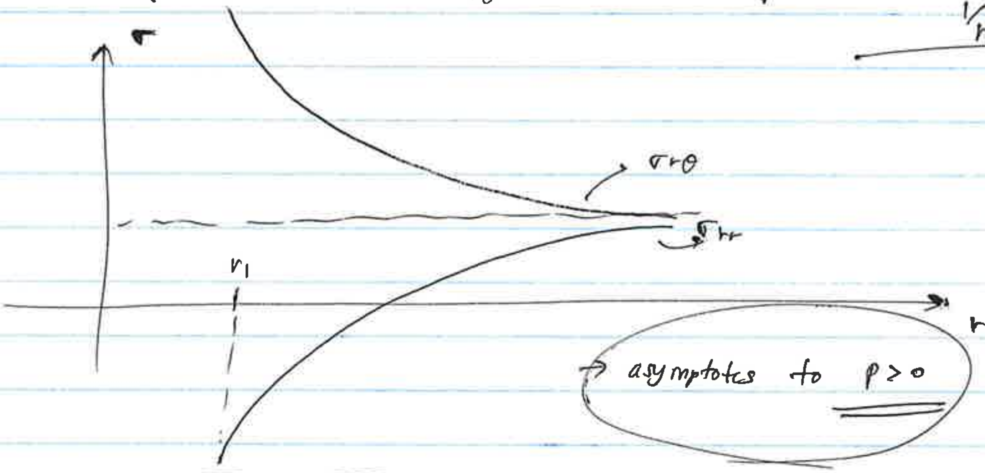


$$\sigma_{rr} = \begin{cases} -p_1 & (r = r_1) \\ -p_2 & (r = r_2) \end{cases}$$

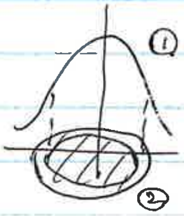
$\sigma_{r\theta} = 0.$ (surf ① & ②)

propose

$$\left.\begin{aligned} \sigma_{rr} &= A - B/r^2 \\ \sigma_{\theta\theta} &= A + B/r^2 \\ \sigma_{r\theta} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} \\ B &= \frac{p_2 - p_1}{1/r_1^2 - 1/r_2^2} \end{aligned}$$



→ asymptotes to $p > 0$



* Check if $r_{①} \simeq r_{②}$ is satisfied.

For thin wall, (vessel)

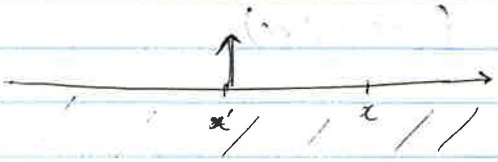
$r_2 = r_1 + t$

$r_2^2 - r_1^2 \sim 2r_1 . t$

$\sigma_{\theta\theta}(r = r_1) = p_1 r_1 / t$

- Contact. Problemm 04/29/2024.

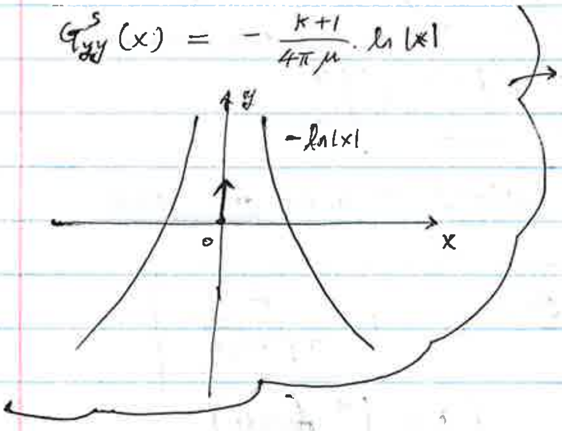
- Surface green function. — George Green (Physics today 1985)



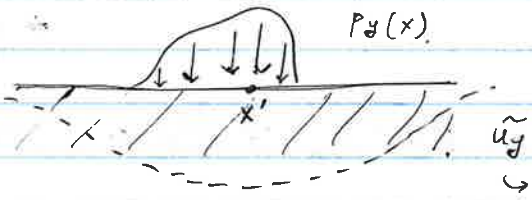
$\underline{G^{S}_{ij}(x, x')}$ (i direction of displacement, j " force.)

* must be translational invariant. (e.g. infinite space).

$$G^{S}_{yy}(x) = -\frac{k+1}{4\pi\mu}\ln|x|$$



Assume only normal forces,

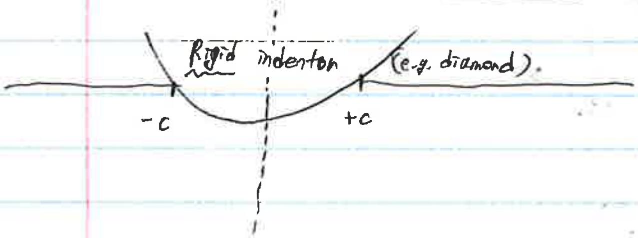


$$\tilde{u}_y(x) = \int_{-\infty}^{+\infty} -P_y(x')\,G^{S}_{yy}(x-x')\,dx'$$

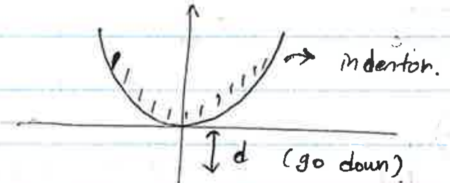
(∵ downward force is defined positive)

$$\Rightarrow \underline{\underline{\tilde{u}_y(x) = \int_{-\infty}^{+\infty} \frac{k+1}{4\pi\mu} P_y(x')\ln(x-x')\,dx'}}$$

- Set up frictionless contact problem.

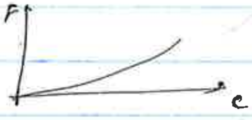


Indentor shape : $\underline{u_0(x)}$



① $u_0(x) - d = \int_{-\infty}^{\infty} P_y(x')\cdot\frac{k+1}{4\pi\mu}\cdot\ln(x-x')\,dx' = \int_{-c}^{+c} ▨ \,dx'$ (no force outside of $[-c, c]$).

② $F = \int_{-c}^{+c} P_y(x')\,dx'$.



B.C. $-c < x < c$ ($y = 0$)	$\lvert x\rvert > c$ ($y = 0$)
$\tilde{u}_y(x) = u_0(x) - d$	$\tilde{u}_y(x) < u_0(x) - d$
$\sigma_{yy}(x) = -P_y(x) < 0$	$\sigma_{yy}(x) = 0$
$\sigma_{xy}(x) = 0$	$\sigma_{xy}(x) = 0$
contact area	gap area

$$u_0(x) - d = \int_{-c}^{+c} P_y(x') \cdot \frac{k+1}{4\pi\mu} \cdot \ln(x-x')\, dx'$$

$$\downarrow$$

$$\underline{\underline{\frac{d u_0(x)}{dx} = \frac{k+1}{4\pi\mu} \cdot \int_{-c}^{+c} \frac{p_y(x')}{x-x'}\, dx'}} \qquad (-c < x, x' < c)$$

Q) can we find f(x) where

$$g(x) = \int_{-c}^{+c} \frac{f(x')}{x-x'}\, dx' \longrightarrow \text{Principal value } (\text{avoid } x = x') \quad (x' = x-\varepsilon, x+\varepsilon)$$

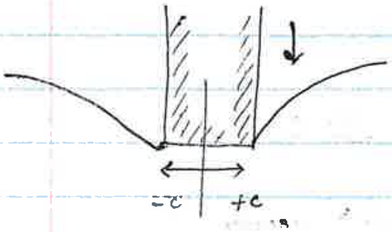
$$\Rightarrow \int_{-c}^{+c} = \int_{-c}^{x-\varepsilon} + \int_{x+\varepsilon}^{c}$$

$$g(x) = \int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{f(x')}{x-x'}\, dx' \quad (\text{Hilbert transform}) \text{ — is its } \underline{\text{own inverse}}$$

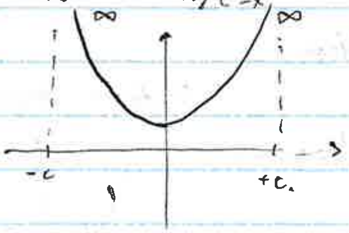
$$\boxed{f(x) = -\frac{1}{\pi^2 \sqrt{c^2-x^2}} \int_{-c}^{+c} \frac{\sqrt{c^2-x'^2}\; g(x')}{x-x'} \cdot dx' + \frac{F}{\pi\sqrt{c^2-x^2}}}$$

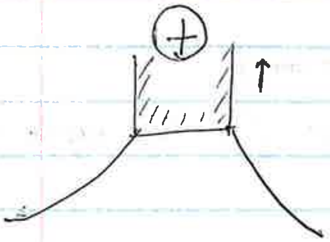
$$\left(F = \int_{-c}^{+c} f(x)\, dx \right)$$

Example 1.

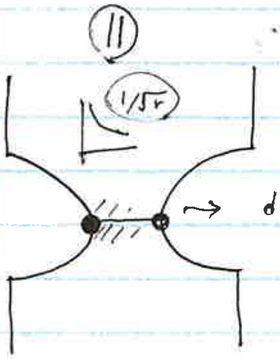


$$P_y(x) = \frac{1}{\pi\sqrt{c^2-x^2}} F$$





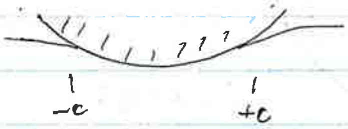
$$\left(\begin{array}{l} x = c - r, \quad r \to 0 \\ \sigma \sim 1/\sqrt{r} \end{array} \right)$$



→ double crack — same stress singularity as crack

- Example 2. – cylindrical punch

$u_0(x) = x^2/2R$



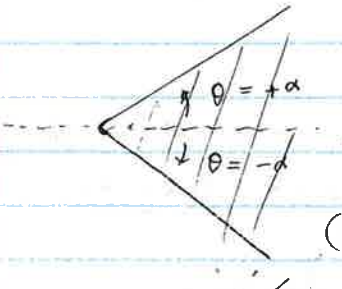
$du_0/dx = x/R$ → plug into integral → get $\underline{p_y(x)}$

→ get where $p_y(x) = 0$ → $\underline{\underline{x = \pm c}}$

$$p_y(x) = \frac{4\mu}{(k+1)R}\cdot\left(\sqrt{c^2-x^2} - \underbrace{\frac{c^2}{2\sqrt{c^2-x^2}}}_{①}\right) + \underbrace{\frac{F}{\pi\sqrt{c^2-x^2}}}_{②}$$

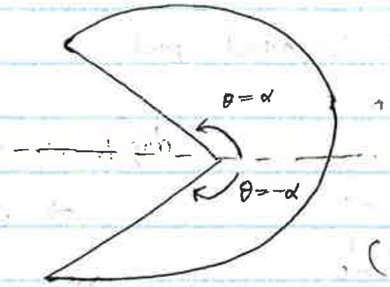
Intuition → no <u>singularity!</u> ①, ② term cancels.

Contact problem ↗ Wedge and Notch. 05/01/2024.



$(2\alpha < \pi)$

<no singularity>



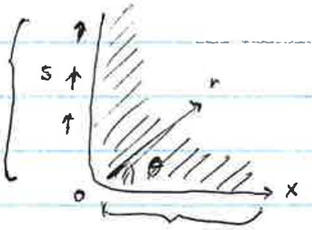
$(2\alpha > \pi)$

<singularity>

Example 1.

B.C.

$$\begin{cases} \sigma_{xy} = -S \\ \sigma_{yx} = 0 \end{cases}$$



$$\text{B.C.} \begin{cases} \sigma_{xy} = 0 \\ \sigma_{yy} = 0 \end{cases} \quad (y = 0)$$

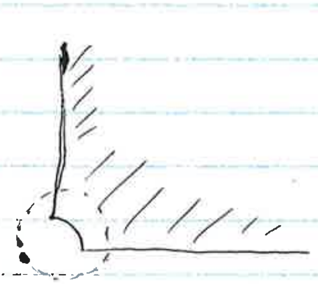
Q) At $x, y = 0, 0$ what is σ_{xy}

Is it $-S$ or 0. $\sigma_{xy} \neq \sigma_{yx} = \phi_{x,y}$

what about $\underline{\underline{\phi_{x,y} \neq \phi_{y,x}}}$?

commutation does not hold.

$$\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x}\phi \neq \frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi\right) \ ?$$



Get rid of "singularity"

$$\begin{cases} \nabla^4 \phi(r,\theta) = 0 \\ \phi(r,\theta) = (r^m) e^{in\theta} \\ (m = n, -n, 2+n, 2-n.) \end{cases}$$

$\sigma_{r\theta} = r^0$, $\phi \sim r^2$ $\left(\because \ \partial^2\phi/\partial r^2 + \sim\sim \right)$

$(m = 2, n = 2, 0)$ $\quad$ " $\sigma_{r\theta} \rightarrow$ second derivative.

$\phi = r^2, \ r^2 \cos 2\theta, \ r^2 \sin 2\theta, \ \underset{\sim}{r^2\theta}$

→ Not in a table. (intuition)

In fact, $\nabla^4 (r^2\theta) = 0$

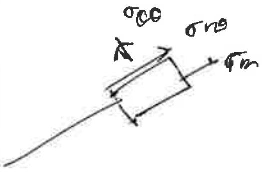
Our own table

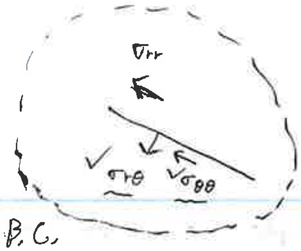
ϕ		$r^2\theta$
σ_{rr}		2θ
$\sigma_{r\theta}$	~~~~	-1
$\sigma_{\theta\theta}$		2θ

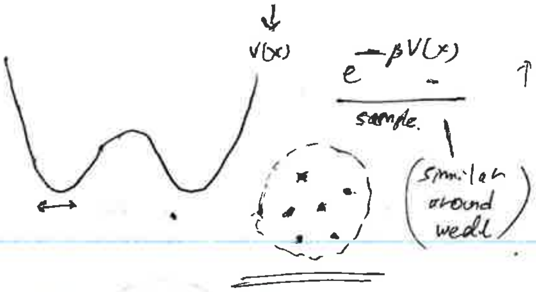
$\Rightarrow \phi =$ ______ (notes)

$\sigma_{xy} =$ ______ (notes)

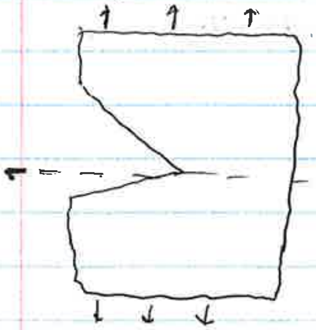
$$\left(\sigma_{xy} = \sigma_{yx} = -\frac{Sy^2}{x^2+y^2}\right)$$







- ~~Wedge~~ Notch.



B.C.

$\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ $(\theta = \pm\alpha)$

Williams Solution

$\phi = r^{n+2}\{A_1\cos(n+2)\theta + A_2\cos n\theta + A_3\sin(n+2)\theta + A_4\sin n\theta\}$

$(n+2 \doteq \lambda+1)$

$\phi = r^{\lambda+1}\{ \sim \}$

$\sigma_{rr} = r^{\lambda-1}\{ \sim \}$ If $\lambda<1$ → stress field is singular

$\sigma_{r\theta} = r^{\lambda-1}\{ \sim \}$

$\sigma_{\theta\theta} = r^{\lambda-1}\{ \sim \}$. Goal : Find λ

Substitute. $\sigma_{r\theta}=0$ $\theta=\alpha$, $\theta=-\alpha$ ⇒ 4 variables. 4 equations.
$\sigma_{\theta\theta}=0$ $\theta=\alpha$. $\theta=-\alpha$.

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} A_1 \\ \vdots \\ A_4 \end{bmatrix} = \vec{0} \Rightarrow \begin{matrix} [M_1]\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \\ [M_2]\begin{pmatrix} A_3 \\ A_4 \end{pmatrix} = 0 \end{matrix} \Rightarrow \begin{matrix} \det|M_1| = 0 \\ \text{or} \\ \det|M_2| = 0 \end{matrix}$$

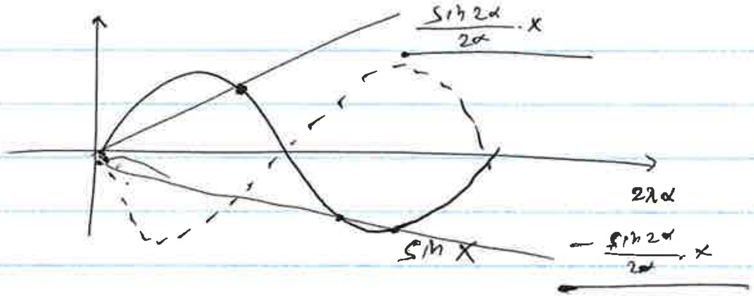
$\Rightarrow \lambda\sin 2\alpha + \sin 2\lambda\alpha = 0$

or

$\lambda\sin 2\alpha - \sin 2\lambda\alpha = 0$.

$x = 2\lambda\alpha$ $\lambda = \frac{x}{2\alpha}$

$$\Rightarrow \boxed{\frac{\sin 2\alpha}{2\alpha}\cdot x \pm \sin x = 0}$$



$\alpha \to \pi$ $\sigma \sim \frac{1}{r}$ $\varepsilon \sim \frac{1}{r}$ → $w = \frac{1}{r^2}$ $\int w\,dr\cdot r > \infty$

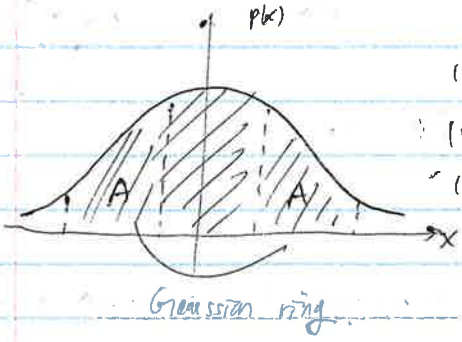
$\lambda = 0, \frac{1}{2}, 1, \ldots$

$\sigma \sim \underbrace{\frac{1}{r}, \frac{1}{\sqrt{r}}}_{\text{singular}}$, , non-singular ⟶ $\sigma \sim \frac{1}{\sqrt{r}}$, $\varepsilon \sim \frac{1}{\sqrt{r}}$ → $w = \frac{1}{r}$ $\int w\cdot r\,dr < \infty$.

$\boxed{\text{finite energy}}$

If B.C. constant force, → infinite energy.

Here B.C. we don't have finite energy. (⇔ $E<\infty$)



(1) Sample from A region

(2) Run M.C. R.W. (obtain length) $\boxed{\langle t_i \rangle}$

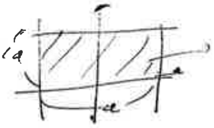
(3) You know $\boxed{p_{s_i}}$

(4) $t = \boxed{\langle t_F \rangle / p_s}$

(5) $\underline{r = 1/t}$?

$k(i \to F) = f(i)$

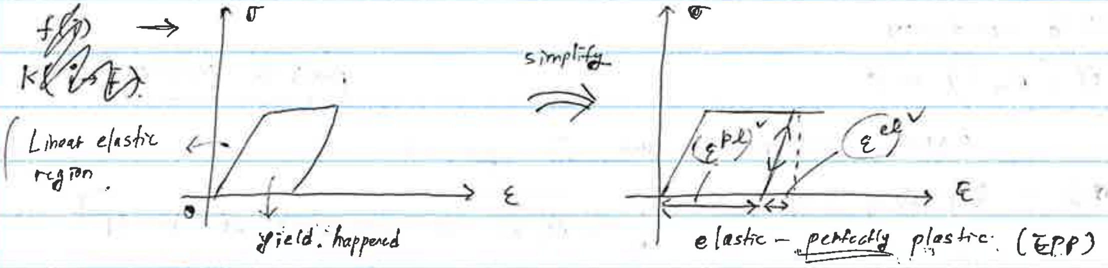
$k(F \to i) = \frac{1}{z} f(i)\, e^{-\beta(V(i) + V(F))}$ =



"uniform"

Plasticity

05/08/2024.



① Displacement field

$\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X} \qquad u_i(X_i)$

② strain field.

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

③ Traction, stress field.

$T_j = \sigma_{ij} \cdot n_i$

④ Equilibrium condition.

$\sigma_{ij,j} + F_j = 0$

⑤ Compatibility condition.

$\varepsilon_{ij,kl} +$ ~~——~~ (automatically satisfied by ②).

✓ $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$ ((compensate each other)

→ satisfies compatibility condition (as a sum).

* Constitutive equation.

- Generalized Hooke's law. ∴ $\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{kl}^{el}$
- Isotropic elasticity : $\sigma_{ij} = \lambda \varepsilon_{kk}^{el} \delta_{ij} + 2\mu \varepsilon_{ij}^{el}$

$\bar{\sigma} = 1/3 \cdot \sigma_{ii} = 1/3(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ → does not change. (hydro-static stress).

$\bar{\varepsilon} = 1/3 \cdot \varepsilon_{ii} = 1/3(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$ → (hydro-static strain)

⇒ $\boxed{\bar{\sigma} = \underset{\text{bulk modulus}}{3k}\, \bar{\varepsilon}^{el}}$

* deviatoric stress : $S_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij}$

" strain : $e_{ij} = \varepsilon_{ij} - \bar{\varepsilon}\delta_{ij}$

$\boxed{S_{ij} = \underset{\text{shear modulus}}{\underset{\sim}{2\mu}}\, e_{ij}^{el}}$

→ changes shape

Strain : $\varepsilon_{ij}^{el} = \underset{\sim}{\bar{\varepsilon}_{ij}^{el}} \delta_{ij} + \underset{\sim}{e_{ij}^{el}}$

Stress : $\sigma_{ij} = \textcircled{3K}\, \bar{\varepsilon}_{ij}^{el} \cdot \delta_{ij} + \textcircled{2\mu} e_{ij}^{el} = \sigma_{ij}$

- Yield condition.

$$\underline{f(\{\sigma_{ij}\}) = 0}$$

↓ assume isotropic. (even after plastic)

$$\sigma'_{ij} = Q_{ip} \cdot Q_{jq} \cdot \sigma_{pq}$$

↳ $f(\{\sigma'_{ij}\}) = 0$

$\boxed{\sigma_1, \sigma_2, \sigma_3 \text{ are eig. values of } \{\sigma_{ij}\}}$ — principle stresses

stress invariants
$$\begin{cases} I_1 = tr(\sigma_{ij}) = \sigma_1 + \sigma_2 + \sigma_3 \\ -I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - (\sigma_{ij})^2) = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \cdots \\ \qquad = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = det.(\sigma_{ij}) = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \end{cases}$$

→ I_1 and I_3's existence implies $I_2 \Leftrightarrow 2(I_2)$? → (No) (each independent)

⇒ ∴ $f(\cancel{I_1}, I_2, I_3) = 0$ ⟶ (plasticity doesn't depend on pressure) → experimental result

$\| \; I_1$

↓

$f(I_2, I_3) = 0$ ⟸ $f(\{s_{ij}\}) = 0$

(∵ s_{ij} also gets rid of I_1).

↓

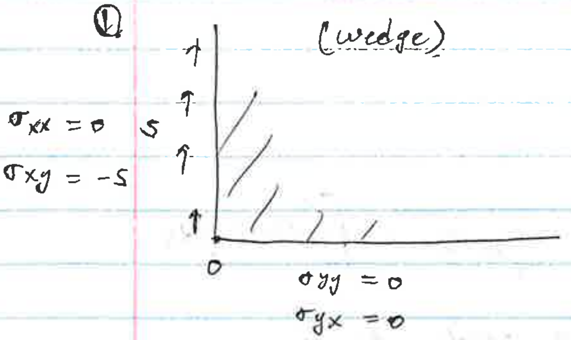
$$\begin{cases} J_1 = tr(s_{ij}) = 0 \\ J_2 = \frac{1}{2}(s_{ij})^2 \\ J_3 = det\{s_{ij}\} \end{cases} \Rightarrow \underline{\underline{f(J_2, J_3) = 0}}$$

⇓

$\boxed{f(J_2) = 0}$ (∵ experiment)

Problem session.

①



polar $\longrightarrow$ ① $\theta = 0$ $\quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0$

② $\theta = \pi/2$ $\quad \sigma_{\theta\theta} = 0 \quad \sigma_{r\theta} = +S$.

$\sigma_{r\theta} \sim r^0$

$\phi \sim r^2 \Rightarrow \boxed{m = 2}$ $\quad \underline{n = \cancel{0}, 2}$

Stress $\sim r^n \Leftrightarrow \phi \sim r^{n+2}$

William's solution states that.,

$$\phi(r,\theta) = r^{n+2}\{A_1 \cos(n+2)\theta + A_2 \cos(n\theta) + A_3 \sin(\overset{n+2}{\cancel{n+2}})\theta + A_4 \sin n\theta\}$$

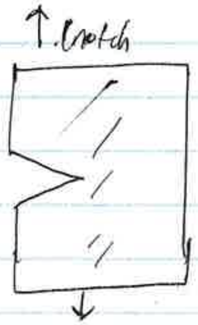
$$\longrightarrow \lambda = n+1 \Rightarrow \phi = r^{\lambda+1}\{A_1 \cos(\lambda+1)\theta + A_2 \cos(\lambda-1)\theta$$
$$+ A_3 \sin(\lambda+1)\theta + A_4 \sin(\lambda-1)\theta.\}$$

→ Take derivative,

$$\sigma_{r\theta} = r^{\lambda-1}[A_1 \lambda(\lambda+1).\sin(\lambda+1)\theta + A_2\lambda(\lambda-1).\sin(\lambda-1)\theta + (-A_3).\lambda(\lambda+1)\cos(1+\lambda)\theta + (-A_4)\lambda(\lambda-1)$$
$$\cos(\lambda-1)\theta]$$

---

②



B.C. $\theta = \pm\alpha$ (traction free)

$\sigma_{r\theta} = 0$, $\sigma_{\theta\theta} = 0$

1) Even in θ (σ_{rr}, $\sigma_{\theta\theta}$) $\Rightarrow$ ϕ is even in θ $\Rightarrow$ $\underline{\sigma_{r\theta} \text{ is odd in } \theta.}$

$\therefore A_3 = A_4 = 0$. → can't say this

$\rightarrow$ plug in $+\alpha \Rightarrow +A_1\,s + A_2 \cdot s. - A_3 \cdot c. - A_4 \cdot c = 0$

" $\quad -\alpha \Rightarrow -A_1\,s - A_2 \cdot s - A_3 \cdot c. - A_4 \cdot c = 0$

$$\Rightarrow \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = 0 \Rightarrow \boxed{\begin{array}{l} |M_1| = 0 \\ |M_2| = 0 \end{array}}$$

To be wedge, $\underline{\pi/2 < \alpha}$

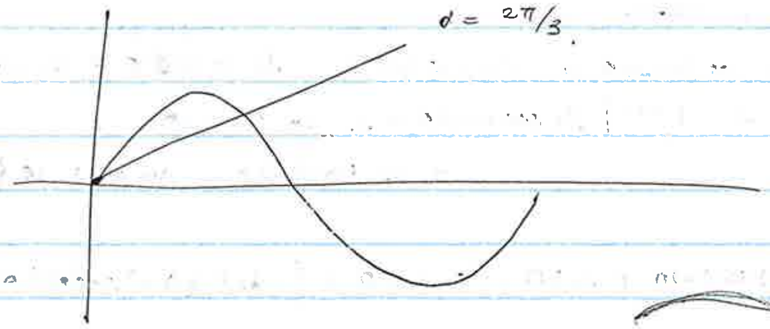
As $\alpha \to \pi$, $\underline{\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots}$

$\boxed{\sigma \sim r^{\lambda-1}}$

$$\int \frac{1}{2} \sigma \varepsilon dr d\alpha \quad \frac{}{r^{2-2\lambda}} = \int \boxed{r^{2\lambda-2}} dr$$

if $\lambda = 0 \to$ <u>explode energy</u>

Also, if $\lambda > 1 \to$ non-singular



$\to$ solve for $\boxed{A_1, A_2} \Rightarrow \boxed{\lambda}$ depends on $\boxed{\alpha}$

$f(\{\sigma_{ij}\}) = 0 \to$ yield criterion

J_1, J_3 are not important (experiment) $\Rightarrow f(J_2) = 0 \ldots$

$I_1 = tr\{\sigma_{ij}\}$

$I_2 =$

$I_3 =$

05/13/2014.

- Plasticity

Basics

$$\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$$

$$\sigma_{ij} = \underbrace{\lambda\, \varepsilon_{kk}^{el} \cdot \delta_{ij}} + \underbrace{2\mu\, \varepsilon_{ij}^{el}}$$

decomposed

$$\boxed{\begin{aligned} &\Rightarrow \sigma_{ij} = \underbrace{\bar{\sigma}\, \delta_{ij}}_{\text{hydrostatic}} + \underbrace{S_{ij}}_{\text{deviatoric}} \\ &\Rightarrow \varepsilon_{ij} = \bar{\varepsilon}\, \delta_{ij} + e_{ij} \end{aligned}}$$

$\rightarrow \bar{\sigma} = \frac{1}{3}\sigma_{kk} = \frac{1}{3} tr(\sigma_{ij})$

$\rightarrow \bar{\varepsilon} = \frac{1}{3}\varepsilon_{kk} = \frac{1}{3} tr(\varepsilon_{ij})$.

where $\underline{\bar{\sigma} = 3k\bar{\varepsilon}}$.

- Yield condition $\Leftrightarrow f(\sigma_{ij}) = 0$. (← assume isotropic).

$\rightarrow f(\sigma_1, \sigma_2, \sigma_3) = 0$ (principale stresses).

$\rightarrow$ stress invariants are,

$$\left.\begin{aligned} I_1 &= Tr(\sigma_{ij}) = \sigma_1 + \sigma_2 + \sigma_3 \\ -I_2 &= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij} \cdot \sigma_{ji}). \\ I_3 &= det(\sigma_{ij}) = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \end{aligned}\right\}$$

$\downarrow$

$f(S_{ij}) = 0$. $f(J_2, J_3) = 0$.

$\downarrow$

$J_1 = Tr(S_{ij}) = 0$

$J_2 = \frac{1}{2} S_{ij} S_{ji}$

$J_3 = det(S_{ij})$.

Von Mises

Example

$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sigma$. $\bar{\sigma} = \frac{1}{3}\sigma$

$S_{ij} = \begin{pmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{pmatrix}$

$\boxed{J_2 = \frac{1}{3}\sigma^2} = k^2$

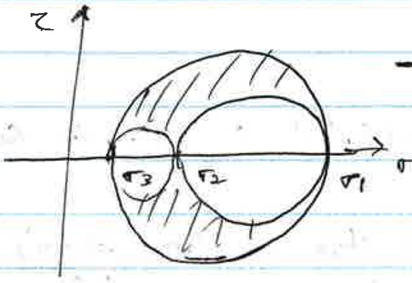
$\rightarrow \boxed{k = \frac{\sigma_Y}{\sqrt{3}}}$

in uniaxial case

- Von-Mises yield condition.

$f(J_2) = J_2 - k^2 = 0$. ∴ yield, $J_2 = k^2$.

- Tressca yield condition.

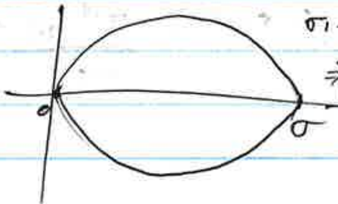


$\sigma_1 - \sigma_3 = 2k_T$ ∴ (w.l.o.g., $\sigma_1 > \sigma_2 > \sigma_3$).

$f((J_2)(J_3)) = 0$ (rewrite).

↳ Messy ...

Example Tressca

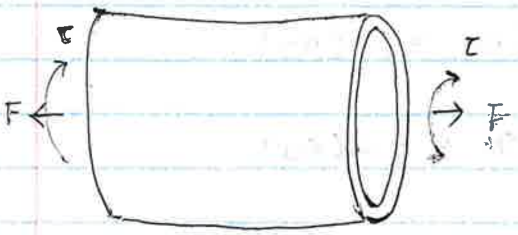


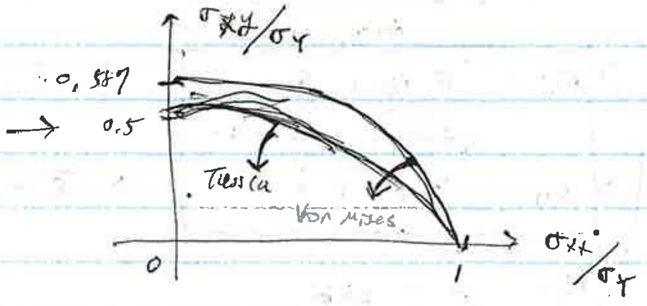
$\sigma_1 - \sigma_3 = \sigma = 2k_T$

$\Rightarrow \boxed{k_T = \frac{\sigma_Y}{2}}$

- How to compare Von-Mises and Tressca ?

→ Taylor & Comy (1931), using tension and shear.





Von-Mises $\begin{pmatrix} \sigma_{xx} & \sigma_{yx} & 0 \\ \sigma_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sigma_{ij} \longrightarrow s_{ij} = \begin{pmatrix} 2/3\,\sigma_{xx} & \sigma_{yx} & 0 \\ \sigma_{xy} & -1/3\,\sigma_{xx} & 0 \\ 0 & 0 & -1/3\,\sigma_{xx} \end{pmatrix}$

$$J_2 = \frac{1}{2}\left(\frac{4}{9}\sigma_{xx}^2 + \frac{1}{9}\sigma_{xx}^2 + \frac{1}{9}\sigma_{xx}^2 + 2\sigma_{xy}^2 \right) = k^2$$

∴ Experimentally, Von-Mises is slightly better.

- Flow rule.

Elastic perfect plastic (EPP)

$J_2 = k^2$ ($\dot{J}_2 = 0$, $\Rightarrow \dot{J}_2 = s_{ij}\dot{s}_{ij} = 0$ (∵ It cannot leave yield surface.)

$\sigma_{ij} = \bar{\sigma}\delta_{ij} + s_{ij}$, $\bar{\sigma} = 3K\bar{\varepsilon}^{el}$, $s_{ij} = 2\mu e_{ij}^{el}$

what is e_{ij}^{pl} ? →, this is path dependent, so only know $\dot{\varepsilon}_{ij}^{pl}$ (incremental).

$\boxed{\dot{\varepsilon}_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} s_{ij}}$ recall $\boxed{e_{ij}^{el} = \frac{1}{2\mu} s_{ij}}$

Difference

rate!

~ Fluid mechanics.

associative flow.

∴ $tr(\dot{\varepsilon}_{ij}^{pl}) = tr(s_{ij}) = 0 \Rightarrow \underline{tr(\varepsilon_{ij}^{pl}) = 0}$ (∵ start from zero)

∴ $e_{ij}^{pl} = \underbrace{\bar{\varepsilon}^{pl}\delta_{ij}}_{=0} + e_{ij}^{pl}$ (no volume change in plastic strain) deformation

$\Rightarrow \boxed{\varepsilon_{ij}^{pl} = e_{ij}^{pl}}$ (only deviatoric strain exists).

05/15/2024.

- Plasticity.

  - Beyond plasticity = <u>Yield criteria</u>.

① Von-Mises.

$J_2 - k^2 = 0$, $J_2 = \frac{1}{2} s_{ij} s_{ij}$. $k = \frac{1}{\sqrt{3}} \sigma_Y$

- Flow rule.

$$\boxed{2\mu \dot{\varepsilon}_{ij}^{pl} = \tilde{\lambda} s_{ij}} ① \longleftrightarrow \boxed{2\mu \dot{e}_{ij}^{el} = \dot{s}_{ij}} ②$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$\dot{\bar{\varepsilon}}^{pl} = 0$ (hydrostatic) $3K \dot{\bar{\varepsilon}}^{el} = \dot{\sigma}$

✓ Note: $\tilde{\lambda} = \frac{2\mu}{2k^2} \dot{W}$ $\left(k = \frac{1}{\sqrt{3}} \sigma_Y\right)$ and $\left(\dot{w} = s_{ij} \dot{e}_{ij}\right)$

Derivation

1) $\dot{w} = s_{ij} \dot{e}_{ij} = s_{ij} \left(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl} \right)$ from ① and ②.

$\Rightarrow 2\mu \dot{w} = s_{ij} \left(2\mu \dot{e}_{ij}^{el} + 2\mu \dot{\varepsilon}_{ij}^{pl} \right)$

$= s_{ij} \left(\dot{s}_{ij} + \tilde{\lambda} s_{ij} \right)$

$\Rightarrow \vec{r} \perp \vec{\dot{r}}$

only at No Hardening
∵ J_2 does not change and
you need to stay on yield surface.

$= \overset{\to 0}{\cancel{s_{ij} \dot{s}_{ij}}} + \tilde{\lambda} s_{ij} \cdot s_{ij} = \tilde{\lambda} \cdot 2 J_2 = 2 \tilde{\lambda} \cdot k^2$

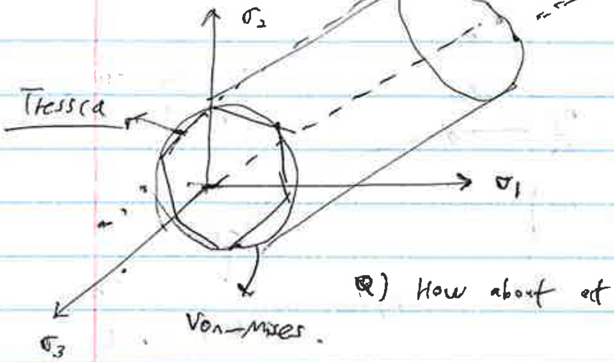
Note : $\dot{w}^{total} = \bar{\sigma} \dot{\bar{\varepsilon}}^{el} + \underset{\sim}{\dot{w}}$

→ deviatoric <u>work</u> done

Idealization : Plastic strain will not cause volume change.

However, if severely deformed, dislocations cause vacancies → <u>Volume decreases</u> ↓

Yield surface in principal stress space.

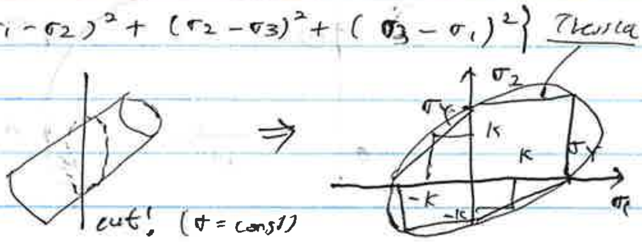


∵ No effect on s_{ij} as σ ↑ (∵ Bridgeman)

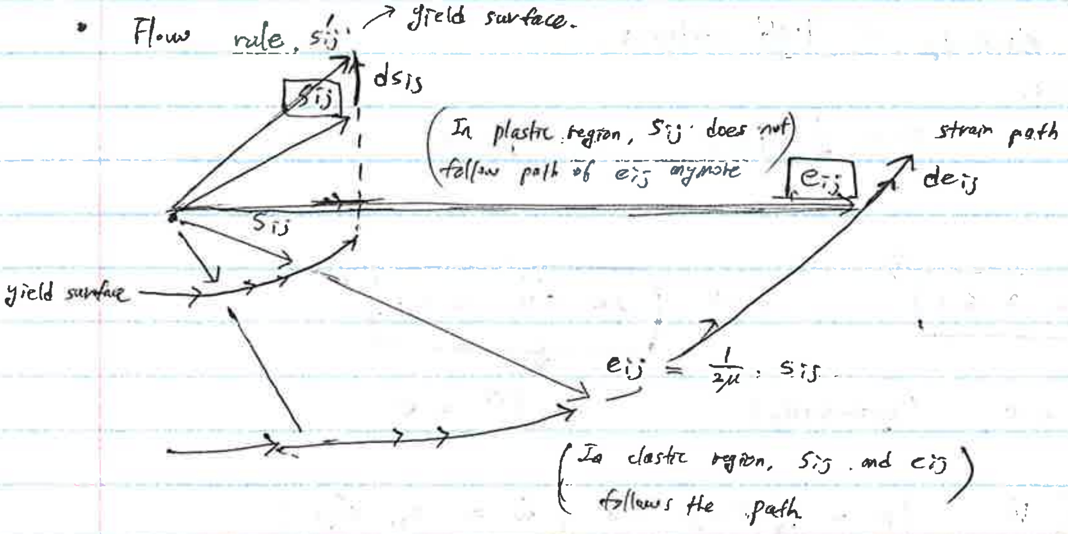
$J_2 = \frac{1}{2} \left(s_1^2 + s_2^2 + s_3^2 \right)$

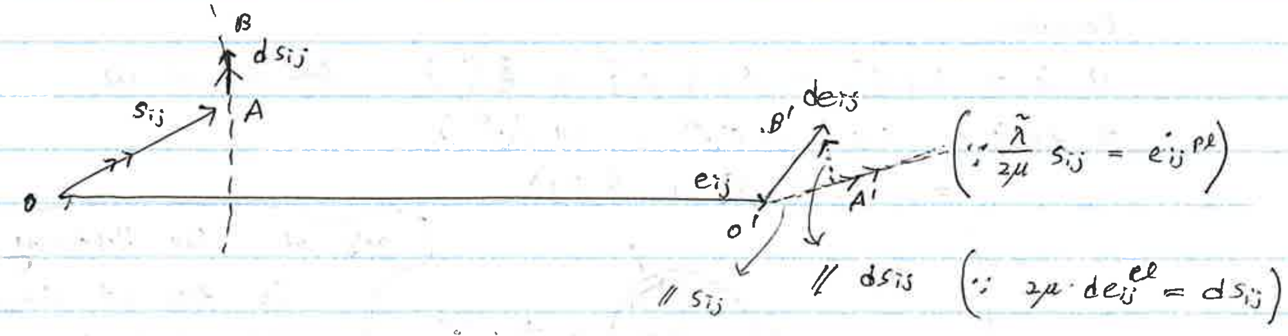
$= \frac{1}{6} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}$

Q) How about at plane stress?



- Flow rule

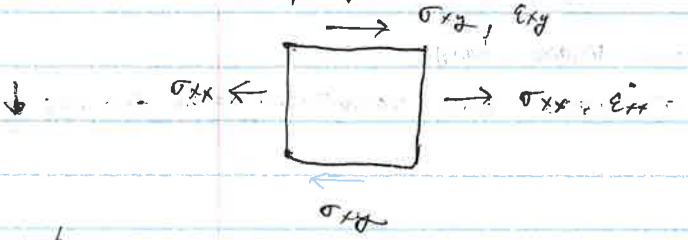




Projection onto the yield surface

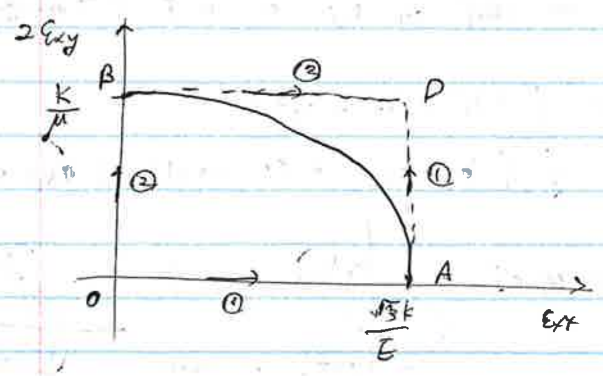
Every time; $e_{ij}^{el}/2\mu$ lies on →

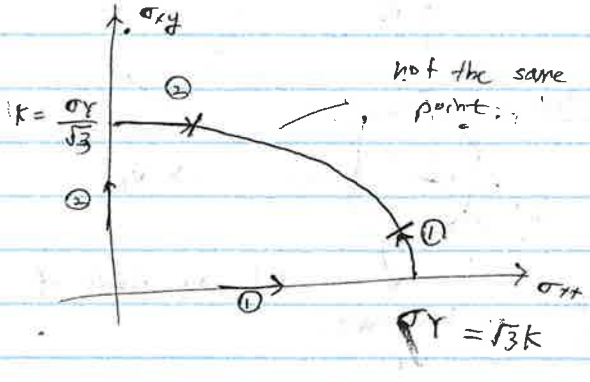
Example 1.



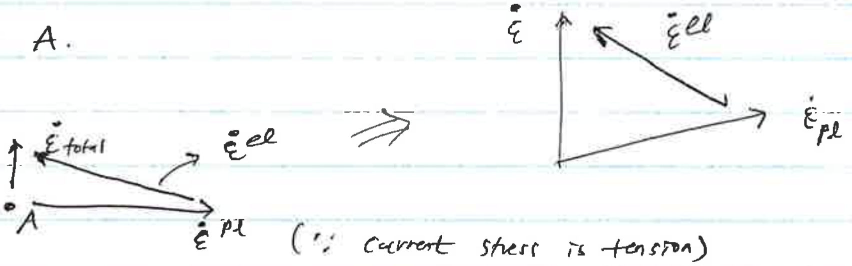
Path ① : $\overrightarrow{OBD}$

Path ② : $\overrightarrow{OAD}$





At point A.



(∵ current stress is tension)

Assume : Incompressible $\nu = 0.5$, $(E = 3\mu)$

$\dot{\varepsilon}_{ij} = \frac{\dot{W}}{2k^2} \cdot s_{ij}$ $\qquad \dot{w} = S_{ij}\, \dot{e}_{ij} = \sigma_{ij}\, \dot{\varepsilon}_{ij}$ (∵ incompressible).

$= (\sigma_{xy}\, 2\dot{\varepsilon}_{xy})$. (∵ path AD)

Recall $J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$, and $\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{xy}^{el} + \dot{\varepsilon}_{xy}^{pl}$

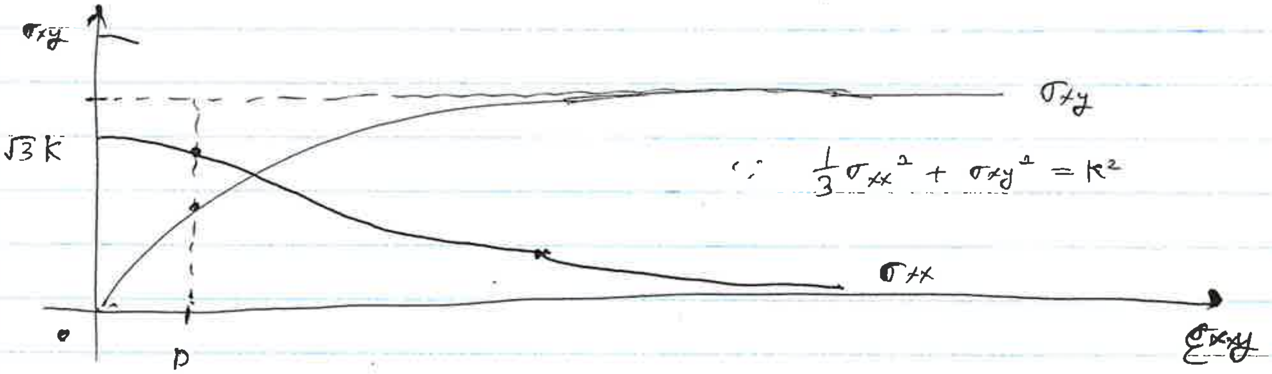
$= \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{w}}{2k^2} \cdot \sigma_{xy}$

$\Rightarrow \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\sigma_{xy}^2 \cdot \dot{\varepsilon}_{xy}}{k^2}$

$\Rightarrow \left(1 - \frac{\sigma_{xy}^2}{k^2}\right) \cdot \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} \Rightarrow \frac{2\mu}{k}\, \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}/k}{1 - \frac{\sigma_{xy}^2}{k^2}}$

$\rightarrow$ Integrate : $2\mu \cdot \frac{\varepsilon_{xy}(t)}{k} = \tanh^{-1}\left\{\frac{\sigma_{xy}(t)}{k}\right\}$

$\Rightarrow \sigma_{xy}(t)/k = \tanh\left(2\mu\, \varepsilon_{xy}(t)/k\right)$.



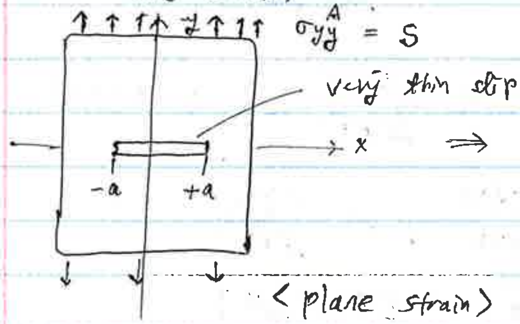
∵ $\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$

(Linear Elastic) plastic

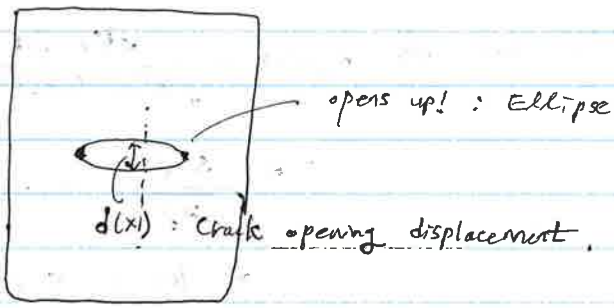
Fracture Mechanics. (LEFM). → EPFM

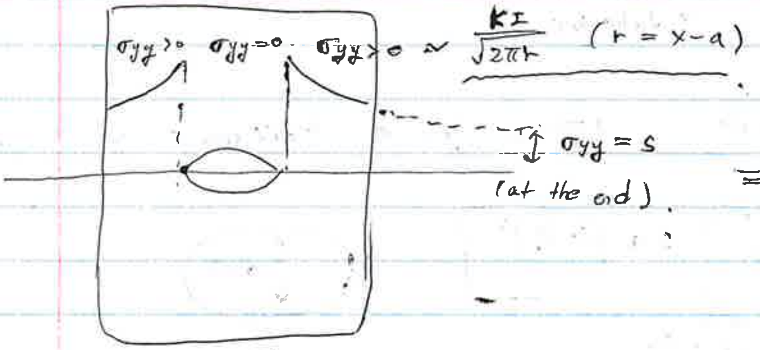
05/20/2024.

- Slit-like crack.



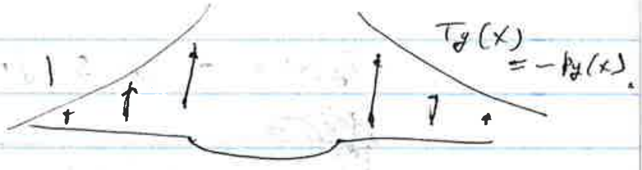
⇒





$\sim \frac{K_I}{\sqrt{2\pi r}}$ $(r = x - a)$

≡



$$\Rightarrow u_y^s(x) = \cancel{\int_{-\infty}^{+\infty}} T_y(x')\cdot -\left(\frac{\kappa+1}{4\pi\mu}\right) \ln(x-x')dx'$$

$$\boxed{\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right)}$$

$$\Rightarrow T_y(x) = \underline{\int_{-\infty}^{-a} + \int_{+a}^{+\infty} \sim} + \frac{Ax+B}{(x+a)^{1/2}(x-a)^{1/2}}$$

($\int \sim$ contains $u_y'(x)$ and in this problem

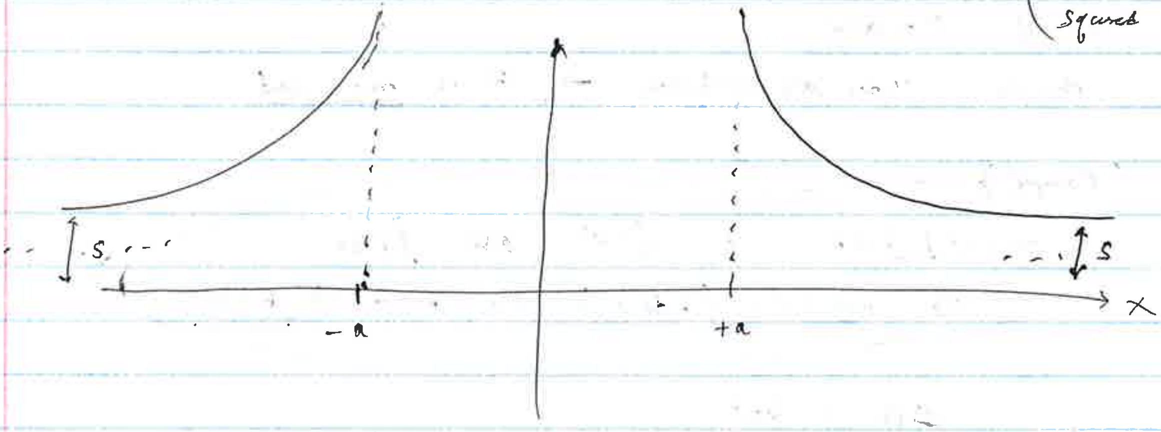
$u_y(x)$ = constant ⇒ $\underline{u_y'(x) = 0}$ at $|x| > a$.)

$$\Rightarrow T_y(x) = \frac{A + B/x}{\sqrt{1-(a/x)^2}}$$ (even function $T_y(x)$ ⇒ $B = 0$)

$$\Rightarrow T_y(x) = \frac{A}{\sqrt{1-(a/x)^2}}$$ ⇒ ($A = S$ ∵ as $x \to \infty$, $T_y(x) = S$)

$$\therefore T_y(x) = \frac{S}{\sqrt{1-(a/x)^2}} \Rightarrow \underset{(y=0)}{\sigma_{yy}(x)} = \frac{S|x|}{\sqrt{x^2-a^2}}$$ (! important !)

($|x|$ is multiplied for squared root term).



- We expect $\sigma_{yy}(r) = \frac{k_I}{\sqrt{2\pi r}}$ and $\sigma_{yy}(x) = \frac{S|x|}{\sqrt{a^2 - x^2}}$

$x = a + r \Rightarrow \sigma_{yy}(x) = \frac{S|(a+r)|}{\sqrt{a^2 - (a+r)^2}} \cong \frac{S \cdot |a|}{\sqrt{2ar}}$ $(\because r \ll 1)$

$$= \frac{S \cdot \sqrt{a}}{\sqrt{2r}} = \frac{S \cdot \sqrt{a\pi}}{\sqrt{2\pi r}} \sim \boxed{\frac{1}{\sqrt{r}}}$$

$$\Rightarrow \underbrace{k_I = S \cdot \sqrt{a\pi}} \quad [Pa \cdot m^{1/2}]$$

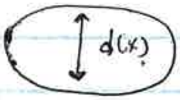
→ stress intensity factor

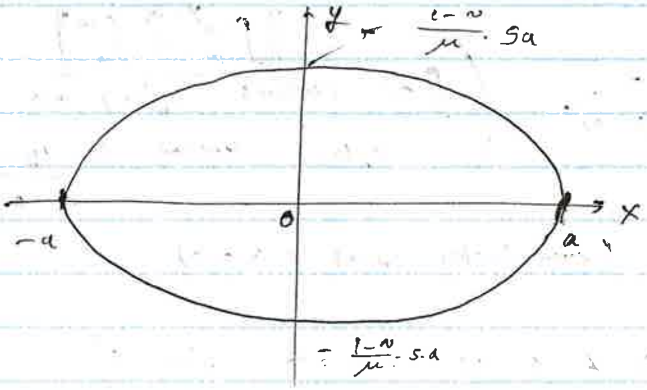
↱ Intuitive.

$\tilde{u}_y(x) = -\frac{k+1}{4\mu} \cdot S \cdot a \sqrt{1 - (x/a)^2}$, $|x| < a$, $y = 0$ (half spae ─◡─)

~~$\frac{d\tilde{u}_y}{dx}$~~ =

$$\boxed{d(x) = -2\tilde{u}_y(x) = \frac{2(1-\nu)}{\mu} S \cdot a \cdot \sqrt{1 - \left(\frac{x}{a}\right)^2}}$$





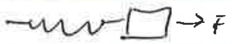
Enthalpy

$E = H + \Delta W$ (We apply work to system the energy increases)

$\Rightarrow H = E - \Delta W$

principle : Under ΔW mechanism, → H is minimized.

(Example)



$E = \frac{1}{2}kx^2$ $\Delta W = \int F dx$

$\Rightarrow \min\left(E - \int F dx\right) \Rightarrow kx - F = 0 \Rightarrow \underline{F = kx}$

$\Delta W = kx^2$

$E = \frac{1}{2}kx^2 \rightarrow H = -\frac{1}{2}kx^2$

I

$$H = E - \Delta W_{LM}$$

$$\begin{cases} E = \int_{\Omega} \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, dV \\ \Delta W = \int_{S_t} T_j u_j \, dS \end{cases}$$

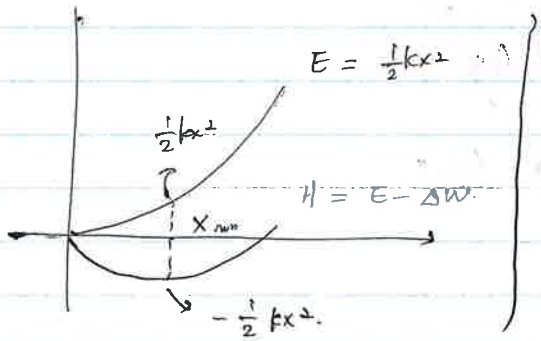
Q) What is Enthalpy of crack?

	system (without crack)	system (with crack)
E	(uniform) $E_0 = \sigma_{ij}^A \varepsilon_{ij}^A \cdot V$	E_1
H	H_0	H_1

$$\left(\begin{array}{l} \Delta E = E_1 - E_0 \\ \Delta H = H_1 - H_0 \end{array} \right).$$

$$\boxed{\begin{array}{l} \Delta H = -\frac{1-\nu}{2\mu} s^2 \pi a^2 < 0 \\ \Delta E = \frac{1-\nu}{2\mu} s^2 \pi a^2 > 0 \end{array}}$$

Energy increases, Enthalpy decreases.



same conditional

Note: $\left(\begin{array}{l} \Delta W = 2 \Delta E, \\ \Delta E - \Delta W = \Delta H \doteq - \Delta E \end{array} \right).$

3D  $R = ?$

$f(i \to j) = 0$ where (2D-LJ)

$i = (x_1, y_1, x_2, y_2)$ and

$j = (x_1, y_1, x_1, y_1)$. — overlap

Fracture Mechanics 05/22/2024.

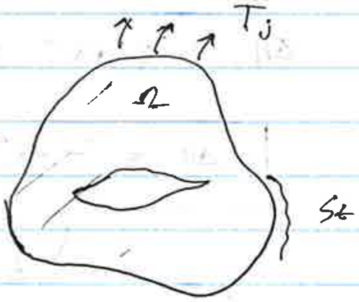
We start from,

$$\left.\begin{aligned} \sigma_{yy}(x) &= \frac{S|x|}{\sqrt{x^2-a^2}} \quad (y=0) \\ d(x) &= \frac{2(1-\nu)}{\mu} S \cdot a \sqrt{1-(a/x)^2} \end{aligned}\right\}$$

- Enthalpy. (H)

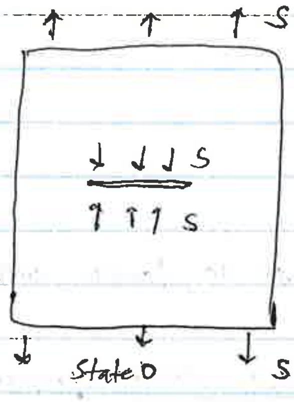
$$H = E - \Delta W_{LM}$$

For linear elastic materials, $\left\{\begin{aligned} E &= \int \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\,dV \\ W &= \int_{St} T_j u_j\,dS. \end{aligned}\right.$

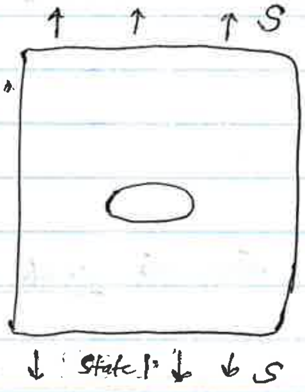


$$\therefore H = \int_\Omega \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} - \int_{St} T_j u_j\,dS$$

Body with no pre-existing internal stress, then $H = -E$.



(prevent from 벌어짐).



$\boxed{E_1} \Rightarrow H = -E_1$

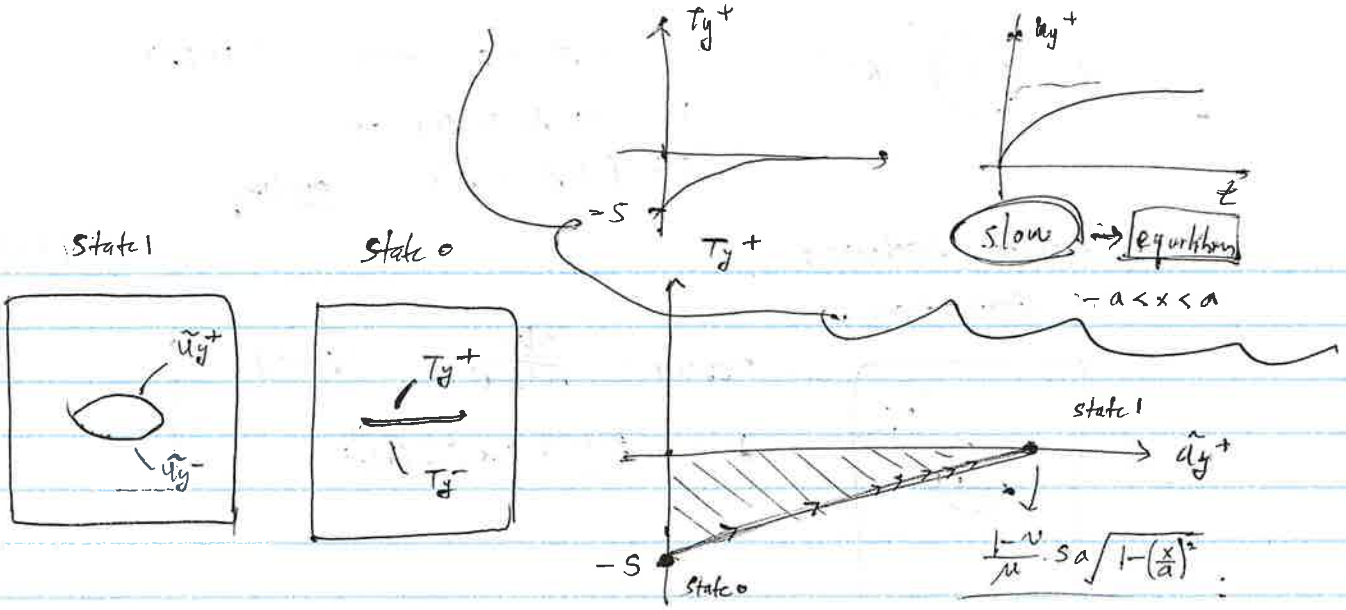
$$E_0 = \tfrac{1}{2}\sigma_{yy}^A \varepsilon_{yy}^A \cdot V$$

$$H = -\tfrac{1}{2}\sigma_{yy}^A \varepsilon_{yy}^A \cdot V.$$

$$\Delta E = E_1 - E_0$$

$$\Delta H = H_1 - H_0 = -\Delta E$$

Can do this because it is ★ path independent — elastic (purely)



$$\Delta W^+ = \int_{-a}^{+a} \frac{1}{2} S \cdot \frac{1-\nu}{\mu} \cdot Sa\sqrt{1-\left(\frac{x}{a}\right)^2}\, dx$$

$$\Delta H = 2\,\Delta W^+ = \int_{-a}^{+a} S\ \frac{1-\nu}{\mu}\ Sa\sqrt{1-(x/a)^2}\, dx.$$

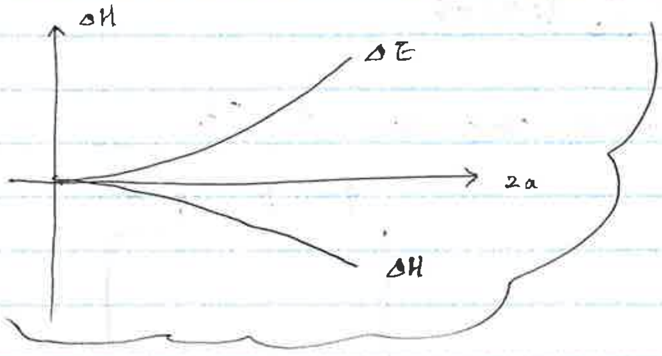
$$\boxed{\therefore \Delta H = -\frac{1-\nu}{2\mu} \cdot S^2 \pi a^2}$$

decreases!

Note:

$$\text{Area} = \frac{1-\nu}{\mu} \cdot S \pi a^2$$

$$\Delta H = A \cdot S/2$$



- Driving force for crack extension. (wants to go to lower enthalpy).

$$f_{cl} = -\frac{\partial(\Delta H)}{\partial(2a)} = \frac{\pi(1-\nu)}{2\mu} \cdot S^2 a.$$

(larger cracks propagate more!)

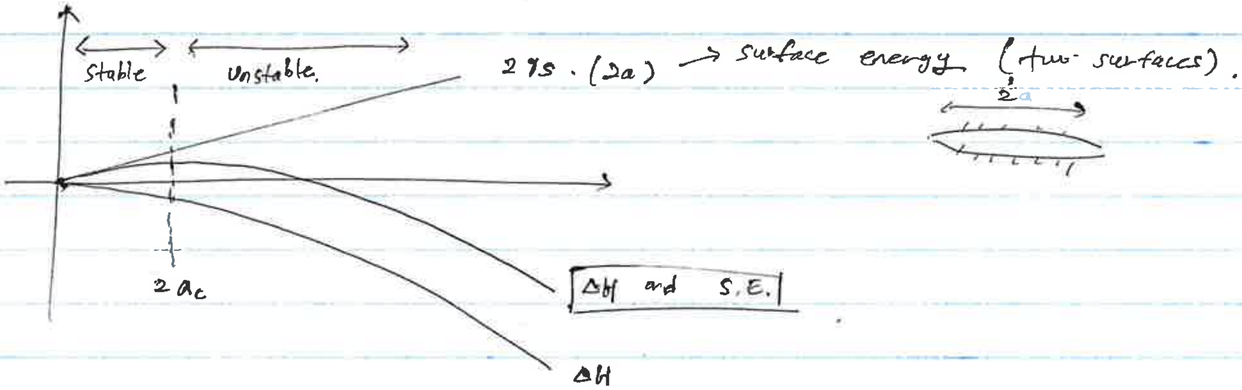
Stronger urge to extend!

$$\Downarrow$$

$$K_I = S\sqrt{\pi a} \quad \rightarrow \quad \boxed{f_{cl} = \frac{1-\nu}{2\mu} \cdot K_I^2}$$

Griffith criteria for brittle materials.

- When you introduce crack, you broke bond (chemical bonds).
  → consider surface energy.



∴ Under certain a, the crack does not propagate

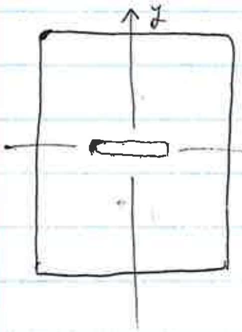
($2a < 2a_c$)

where $\left(2a_c = \frac{8\mu}{\pi(1-\nu)} \cdot \frac{\gamma_s}{S^2} \right)$.

also, $S_0 = \sqrt{\frac{8\mu\gamma_s}{\pi(1-\nu)\cdot(2a)}}$ (critical stress when crack grows).

- Linear – Elastic Fracture Mechanics (LEFM) 05/29/2024.

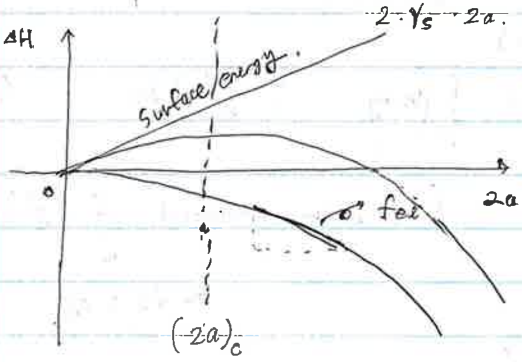
Q) $\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \\ \sigma_{xy} & \sigma_{yy}=0 & \\ & & \sigma_{zz}=0 \end{bmatrix}$

- 

$\Delta H = H_1($  $) - H_0($  $)$

$$= -\frac{1-\nu}{2\mu}\cdot s^2\pi a^2 \quad (\text{plane strain}).$$

$$= -\frac{k+1}{8\mu} s^2\pi a^2 \quad (\text{plane stress}).$$



$$f_{el} = -\frac{\partial \Delta H}{\partial(2a)} = \frac{\pi(1-\nu)}{2\mu} s^2 a \quad (\text{plane strain})$$

→ As crack grows, it grows more!

(∵ |slope| increases as crack ↑)

$$\Delta G = \Delta H + 2\cdot\gamma_s\cdot(2a).$$

$(2a)_c$ is where $\partial\Delta G/\partial(2a) = 0$ (critical point, cross → advances!).

$$\underset{\sim\sim\sim}{\frac{\pi(1-\nu)}{2\mu}s^2\cdot a \geq 2\gamma_s} \longrightarrow \boxed{\text{Griffith criteria}}$$

$2+2+2$ $r_1 = (2,0)$

$r_2 = (1,\sqrt{3})$

$r_3 = (-1,\sqrt{3})$

$$\underline{\sum_i \|\vec{r}_i\|^2 + \sum_{i,j}\vec{r}_i\cdot\vec{r}_j}$$

$r_1 = (1,\sqrt{3})$

$2+-2+2$ $r_2 = (2,0)$

$r_3 = (1,-\sqrt{3})$

- G (Energy release rate) $= f_{el} = -\frac{\partial}{\partial(2a)}\Delta H$.

$\Downarrow$ ≥

G_c (Critical energy release rate) $= 2\gamma_s$

$$G = \frac{\pi(1-\nu)}{2\mu}(\sigma_{yy}^A)^2\cdot a \quad \text{vs} \quad G_c = 2\gamma_s$$

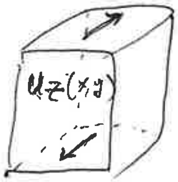
$$\| $$

$$\frac{\pi}{E'}(\sigma_{yy}^A)^2\cdot a \quad (\text{plane strain}), \quad E' = \frac{E}{1-\nu^2}$$

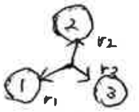
$$\underset{\sim\sim\sim}{K_I = \sigma_{yy}^A\sqrt{\pi a}} \quad (\text{Stresses intensity factor}).$$

$$\Rightarrow \quad G = K_I^2/E' \quad (\text{mode-I loading}).$$

$\Delta G = \Delta H - T\Delta S$

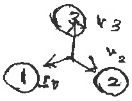


: mode III.

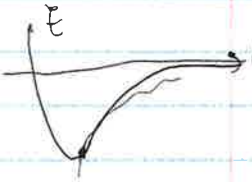


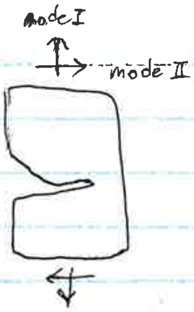
$\|R_{\frac{2\pi}{3}}\vec{r_1} + \vec{r_2}\|^2$

$\vec{r_{12}} \cdot \vec{r_{23}}$



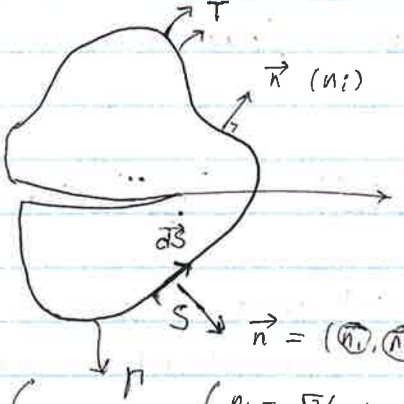
$\Delta S = \int \frac{\delta Q}{T}$





$$\boxed{G = K_I^2/E' + K_{II}^2/E' + K_{III}^2/2\mu}$$
$$\boxed{G \geq G_c}$$

- J - integral



$$J_i = \int_S (\underset{\frac{1}{2}\sigma_{ij}\varepsilon_{ij}}{w}\, n_i - T_j\, u_{j,i})\, dS$$

(In 2D,)

$$J = \int_\Gamma \left(W\, dy - \underset{\sim}{T} \frac{\partial u}{\partial x} \right) dS. \qquad (J_1)$$

(x, y are orthogonal $\vec{n}_x = \vec{y}$)

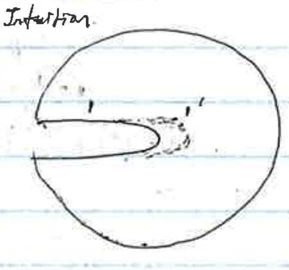
$n_1 dS = dy$

$\vec{n} = (n_1, n_2)$

$$\left(\begin{array}{ll} n_1 = \sqrt{2}/2 & (\theta = -45°) \\ n_2 = -\sqrt{2}/2 & \end{array} \right)$$

$$\Rightarrow \vec{n}\, ds = \left(\underbrace{\frac{\sqrt{2}}{2} ds}_{dx}, \underbrace{-\frac{\sqrt{2}}{2} ds}_{dy} \right)$$

Intuition



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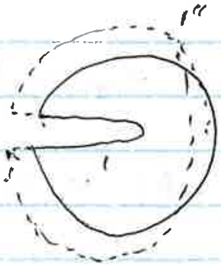


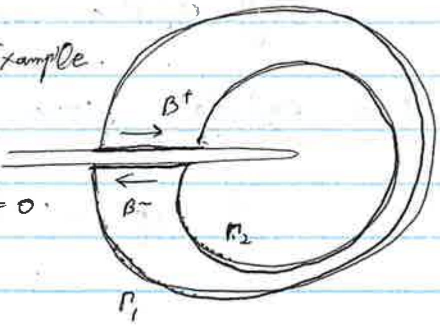
move crack: $\Delta H_{1'} - \Delta H_1$

move contour: $\Delta H_{1''} - \Delta H_1$

Example.

$$J(\Gamma) = J(\Gamma_1) + J(B^+) + J(B^-) - J(\Gamma) = 0$$

(∵ does not contain singularity)



$J(B) = J(B^+) = 0$

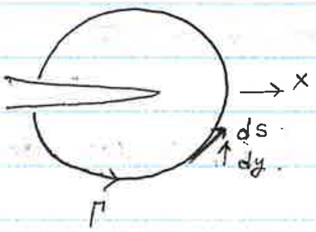
$\therefore J(\Gamma_1) = J(\Gamma_2)$

J - integral.

05/31/2024.

$J = G$ if $G \geq G_c$ (propagates).

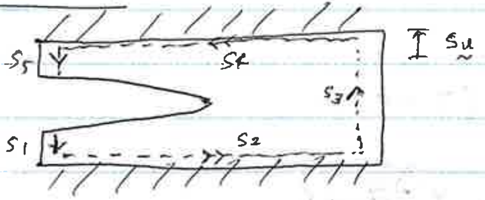
⟨2D⟩



$$J_x = \int_{\Gamma} w \cdot dy - \underset{\sim}{T} \cdot \frac{\partial \underset{\sim}{u}}{\partial x} \cdot ds.$$

(it can be both linear / non-linear).

Example 1.



$u_x = 0$, $u_y = $ constant.

$S_1 \sim S_5$.

$$J(S_2) = \int_1 w \cdot \cancelto{0}{dy} - T \cdot \cancelto{0}{\partial u / \partial x} \cdot ds$$ ———— $(u(x) = const)$.

Similarly, $J(S_4) = 0$

$$J(S_1) = \int \cancelto{0}{w}\, dy - \cancelto{0}{T}\, \cancelto{0}{\partial u / \partial x} \cdot ds.$$

Similarly $J(S_5) = 0$

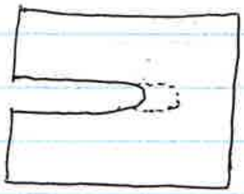
$$J(S_3) = \underbrace{\int w\, dy} - \cancelto{0}{T}\, d\, \underset{\sim}{u} / \partial x \;\; ds$$

Same as when you don't have a crack. $= \boxed{wh}$

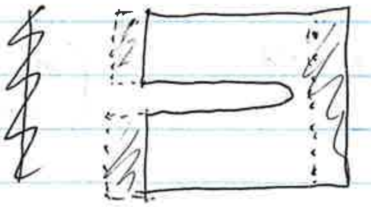
(w is uniform since it's far away).

$\therefore J = w \cdot h.$

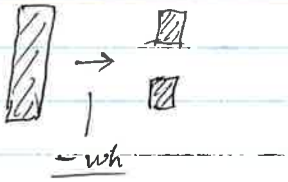
Interpretation why wh is a driving force of a crack.



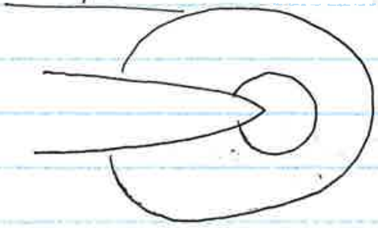
same $\equiv$



we lose pillar of w·h



Example 2.



$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \cdot \left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\sin\frac{3}{2}\theta\right)$$

As $r\downarrow$, σ_{rr}. dominates $\sim 1/\sqrt{r}$

→ Do J-integral.

$\Rightarrow$ $\underset{\sim\sim\sim}{J = K_I^2/E'}$ (singular term dominates).

→ Consistent.

Example 3. Blunted crack tip.

right along the edge. ($T = 0$).

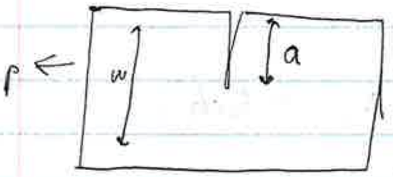
$$J = \int_\Gamma W\, dy - \cancel{T}\overset{0}{\cdot}\frac{\partial u}{\partial x}\, ds.$$

$$= \int_{\Gamma'} W\, dy.$$

↳ we can solve with F.E.M.

- LEFM

$K_I \geq K_{Ic}$



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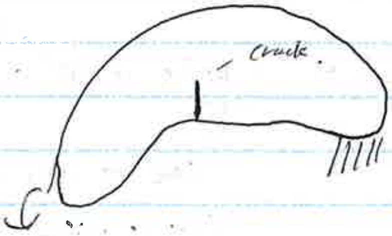
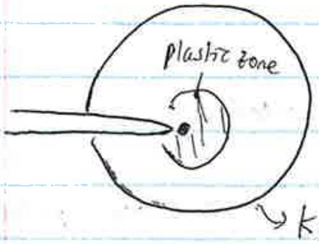


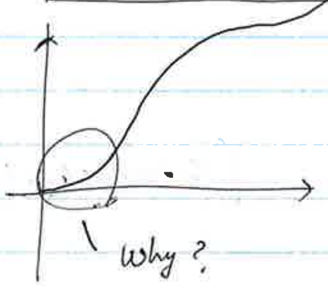


$\sigma \sim K_I/\sqrt{2\pi r}$ — Dominates ✓ non-singular fields

$\boxed{\text{if } K \text{ field exists}}$

(and $\sigma \sim K_I/\sqrt{2\pi r}$ is 90% of total stress)

IJSS – hyper-elastic – collagen

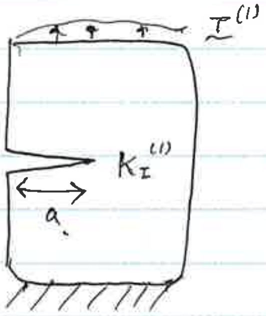


$n_\ell \sim \exp(-\beta \varepsilon_\ell)$

if $n_0 = n_0$ (you fix) $=$ Const. $\cdot \exp(-\beta\varepsilon_0)$

$\Rightarrow n_\ell = n_0 \cdot \exp(\beta\varepsilon_0) \cdot \exp(-\beta\varepsilon_\ell)$.

- Rice (1972) – How to solve $\boxed{K_I}$ for arbitrary loading ?



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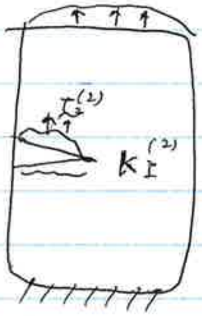
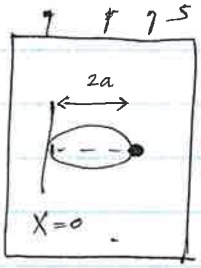


<< Loading 2 >>

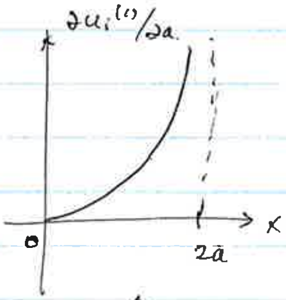
$$K_I^{(2)} = \frac{E'}{2K_I^{(1)}} \cdot \int T_i^{(2)} \frac{\partial u_i^{(1)}}{\partial a} \, d\Gamma$$

Example

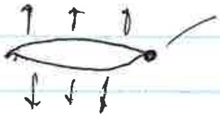
(1)



We know that $\partial u_i^{(1)}/\partial a \propto \sqrt{\frac{x}{2a-x}}$ →



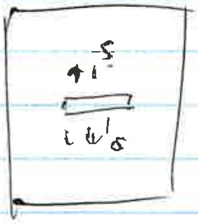
(1+2)



if force is applied close to x=2a, large effect

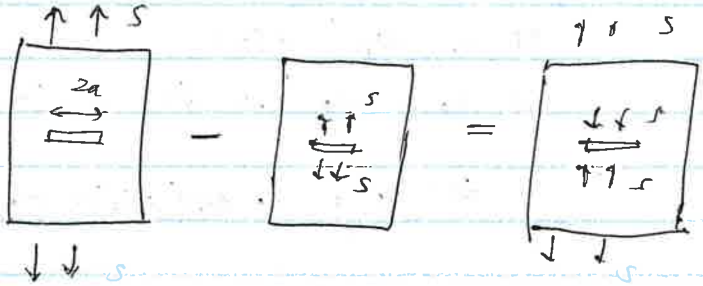
⇒ corresponds to solution $\partial u_i^{(1)}/\partial a$ of (1)

(2)



$$K_I^{(2)} = \sim = \left(\int S \sqrt{\frac{x}{2a-x}}\, dx\right) \cdot \frac{E'}{2K_I^{(1)}} = \boxed{S\sqrt{\pi a}}$$

⇒

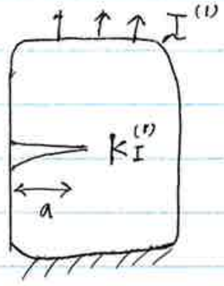


(Interpretation).

$$k_I^{(1)} = S\sqrt{\pi a} - k_I^{(2)} = S\sqrt{\pi a} = k_I^{(3)} = 0$$

- Fracture Mechanics. 06/03/2024.
- LEFM: ($K_I \geq \boxed{K_{Ic}}$) material property.

Example 1.



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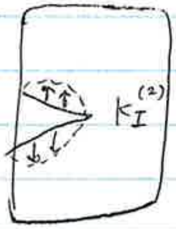

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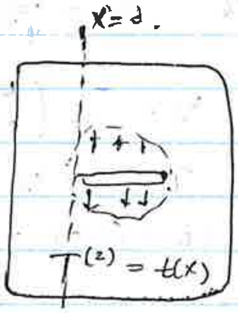
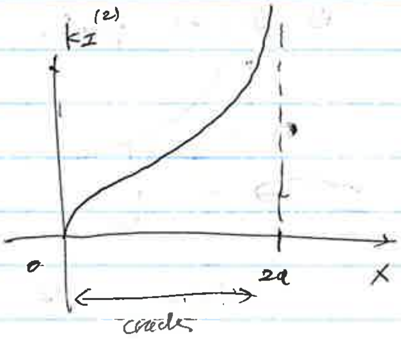
$$\underline{K_I^{(2)} = \frac{E'}{2K_I^{(1)}} \cdot \int_{\Gamma} T_i^{(2)} \frac{\partial u_i^{(1)}}{\partial a} d\Gamma.} \;//$$

Example 2.

$K_I^{(1)} = S\sqrt{\pi a}.$



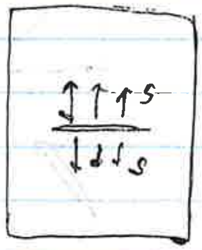


$$\hookrightarrow K_I^{(2)} \doteq \frac{1}{\sqrt{\pi a}} \int_0^{2a} t(x) \sqrt{\frac{x}{2a-x}}\, dx \quad \boxed{(\text{for } x = 2a)} \; ☆$$

↘ Dealing with K_I at $\underline{\underline{x = 2a}}$ (crack: [0, 2a])

(Also, [−a, +a] crack → $K_I^{(2)} = \frac{1}{\sqrt{\pi a}} \cdot \int_{-a}^{+a} t(x) \sqrt{\frac{a+x}{a-x}}\, dx$ (for $x = a$))

Example 3.

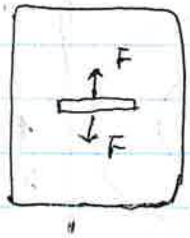


$$K_I^{(3)} = \frac{1}{\sqrt{\pi a}} \cdot \int_{-a}^{+a} S \sqrt{\frac{a+x}{a-x}}\, dx = \boxed{S\sqrt{\pi a}}$$

$$K_I^{(1)} - K_I^{(3)} = S\sqrt{\pi a} - S\sqrt{\pi a} = 0.$$

Superposition!

Example 4.



$$K_I^{(4)} = \frac{E'}{\sqrt{\pi a}} \int_{-a}^{+a} F\, \boxed{\delta(x)} \sqrt{\frac{a+x}{a-x}}\, dx$$

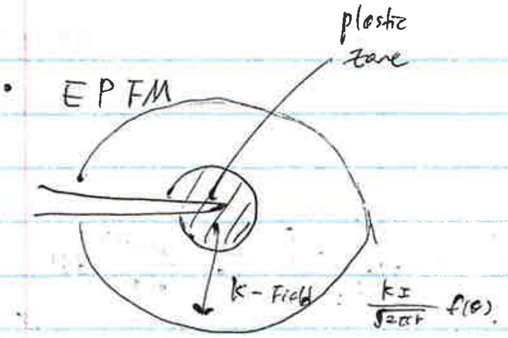
$$= E' F / \sqrt{\pi a}.$$ (reciprocity btw. F and S)

F: force dipole

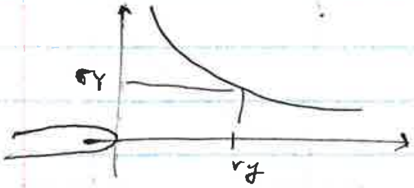
S: stress

- EPFM

Q.) what if "plastic zone is" large?



① Irwin's approach.



Estimate : $\sigma_Y = \frac{K_I}{\sqrt{2\pi \cdot r_y}}$

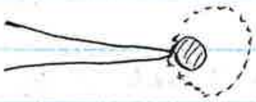
$$\Rightarrow r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$$

$$r_p = 2 \cdot r_y = \frac{1}{\pi}\cdot\left(\frac{K_I}{\sigma_Y}\right)^2$$

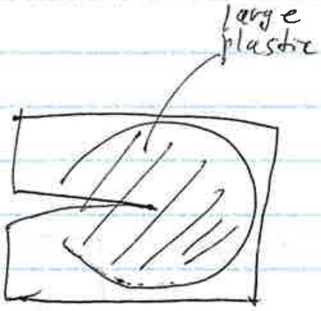
claim : If $r_p < r_k$ (k-field) → Good!

$K_I \gtrsim K_{Ic}$ (σ_Y effect) → yield increase K_{Ic}

$$\boxed{a_{eff} = a + r_y}$$



$$K_{I\,eff} = \frac{P}{B\sqrt{W}} \cdot f\left(\frac{a_{eff}}{W}\right)$$



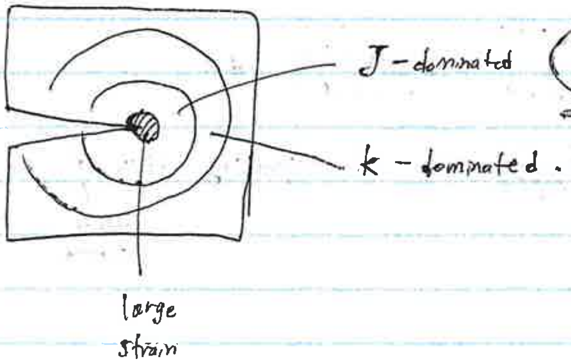
LEFM X

J-integral Y

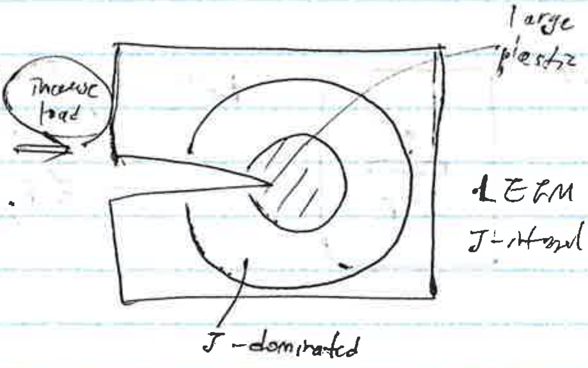
② HRR solution (1968).

$$\varepsilon/\varepsilon_0 = \sigma/\sigma_0 + \alpha(\sigma/\sigma_0)^n$$

$$\Rightarrow \left.\begin{aligned}\sigma_{ij} &= k_1 (J/r)^{\frac{1}{n+1}} \\ \varepsilon_{ij} &= k_2 (J/r)^{\frac{n}{n+1}}\end{aligned}\right] \quad \underline{\sigma\cdot\varepsilon \propto \frac{1}{r}}$$



LEFM ○ J-integral ○



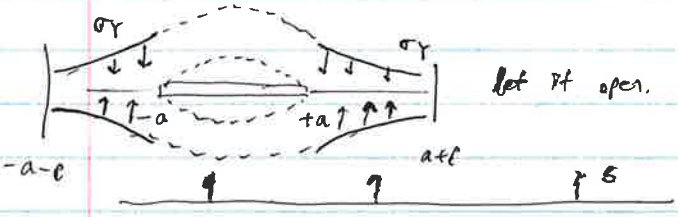
(regions increase →

LEFM X

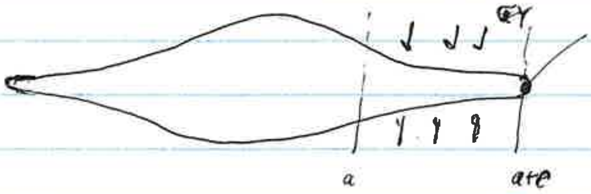
J-integral o

Pb). ① Can't sample
② Too large domain
③ Low temperature.) →

- Strip - Yield model (max stress will be $\boxed{\sigma_Y}$).

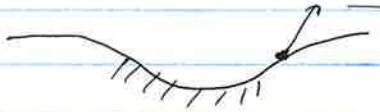


Let it open, with $\boxed{\sigma_Y}$ force, (to close it with yield stress).



$K_{tot} = 0$

In flat punch.



continuous derivative $\longrightarrow k_I = 0$.

$$k_I^{tot} = S\sqrt{\pi(a+e)} \underbrace{- 2\sigma_Y \sqrt{\frac{a+e}{\pi}} \cdot \cos^{-1}\left(\frac{a}{a+e}\right)} = 0$$

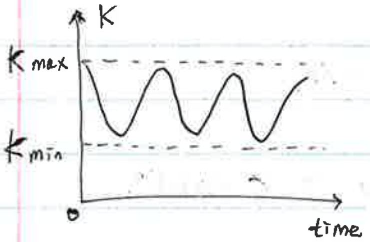
$$= \int_{-a-e}^{-a} + \int_{a}^{a+e}$$

$$\Rightarrow e = \underset{0.393}{\boxed{\frac{\pi}{8}}}\left(\frac{k_{I\,old}}{\sigma_Y}\right)^2$$

(Irwin's $1/\pi$ was 0.318).

- Fatigue. (for now, only LEFM). 06/05/2024.

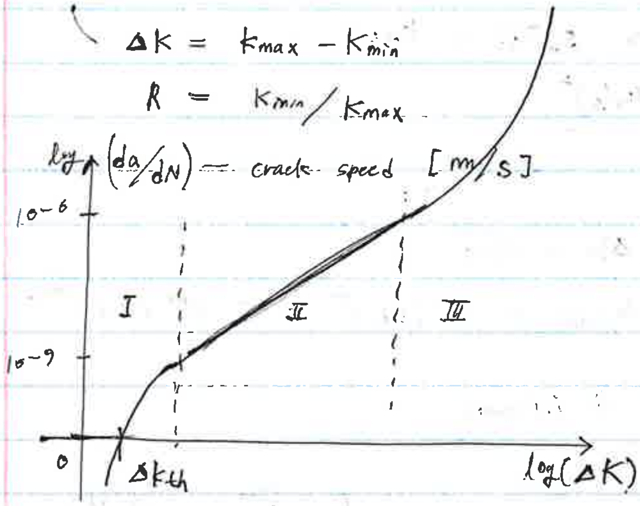
- Paris law.



It doesn't really depend on time, rather depends on cyclic. (cycles).

$\Delta K = K_{max} - K_{min}$

$R = K_{min} / K_{max}$



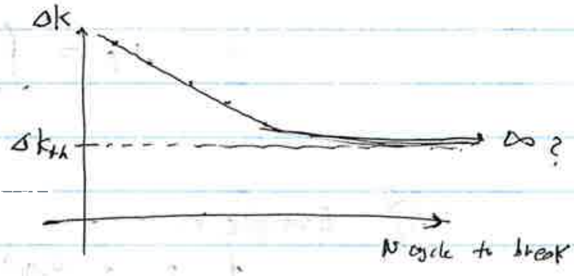
$da/dN = f(\Delta K, R) \cong C (\Delta K)^m$

power law. $2 \leq m \leq 4$ (usually)

These are all empirical.

III : when ΔK too big, it is fine to diverge

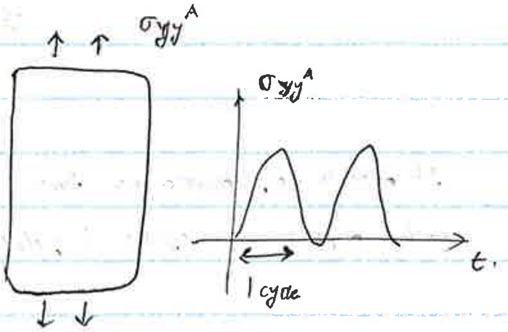
I : At small ΔK values. (threshold region)



$$da/dN = C (\Delta K)^m \frac{\left(1 - \frac{K_{th}}{\Delta K}\right)^p}{\left(1 - \frac{K_{max}}{K_c}\right)^q}$$ → captures blow ups

C, m, ΔK_{th}, K_c, p. q → $\boxed{\text{material property}}$ ✓

Example 1. How many cycle til' fracture.



$k = \sigma_{yy}^A \sqrt{\pi a}$

$k_{max} = S\sqrt{\pi a} = \Delta k$

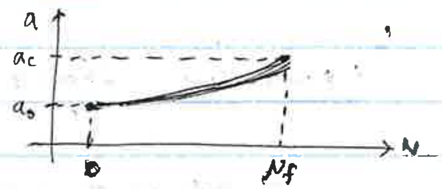
$k_{min} = 0$

assume $da/dN = C \cdot (\Delta k)^m$.

- Critical a ⇒ $k = k_{IC}$ ⇒ $S\sqrt{\pi a_c} = k_{IC}$

$\Rightarrow \underline{a_c = 1/\pi \cdot (k_{IC}/S)^2}$

- Plot a – N



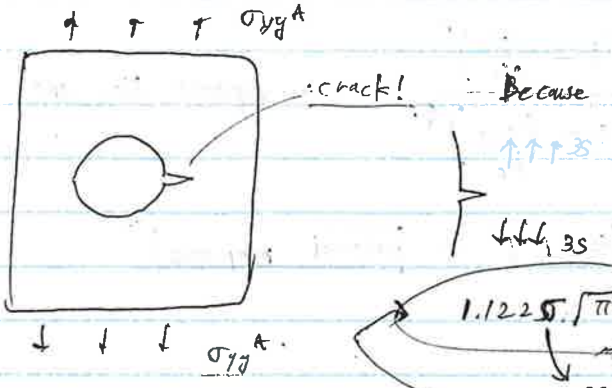
$\Delta k = S\sqrt{\pi a}$

$da/dN = C \cdot (\Delta k)^m = C \cdot S^m (\pi a)^{m/2}$

⇒ Integrate

$$\Rightarrow N_f = \int_{a_0}^{a_c} \frac{1}{C S^m \pi^{m/2}} a^{-m/2} \, da = \left(a_c^{-\frac{m}{2}+1} - a_0^{-\frac{m}{2}+1}\right) \frac{1}{C S^m \pi^{m/2}} \left(\frac{1}{1-m/2}\right)$$

Example 2.



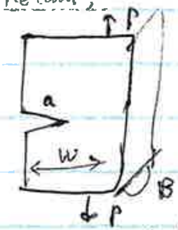
Because of circle, 3 times intensity!

$1.122 S \cdot \sqrt{\pi a} = k$

3S.

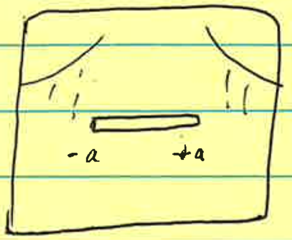
$\Rightarrow \boxed{k = 1.122 \cdot 3S\sqrt{\pi a}}$

Recall.



$$k_I = \frac{P}{B\sqrt{W}}\left(\frac{\sqrt{2\tan(\pi a/2W)}}{\cos(\pi a/2W)}\left(0.752 + 2.02\, a/W\right)\right) = \boxed{P/(BW)\sqrt{\pi a}\,(1.122)}\left(\overset{\infty}{W} \gg a\right)$$

Problemm Session (ME340) 06/07/2024.



slit-like crack

$$g(x) = \int_{-\infty}^{\infty} \frac{p(x')}{x-x'}\,dx'$$

$$\downarrow$$

$$\frac{du(x)}{dx}.$$

$$p(x) = \frac{S|x|}{\sqrt{x^2-a^2}}$$

$$x \to a+r \Leftrightarrow \boxed{\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}}$$

Then, $\boxed{d(x)} = B_{-\infty}\ \frac{2(1-\nu)}{\mu} Sa \cdot \sqrt{1-(x/a)^2} \cdot (-a \le x \le a).$

$\hookrightarrow$ Enthaltpy : In the direction of reducing free energy $(T=0) \to$ Reduce entholpy in any mechanical system.

$$H = E - \Delta W_{LM} = -E \quad \text{(only for Linear Elastic medium)}.$$

Example : $E = \frac{1}{2}kx^2$, $W = kx \cdot x = Kx^2$

state 0

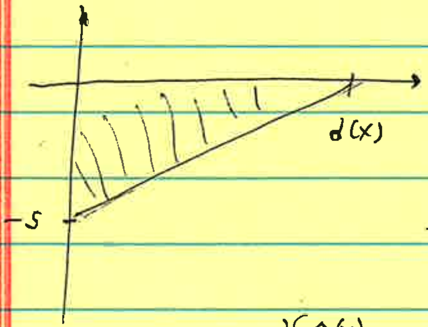
E_0, H_0.

$H_0 = -E_0$

State 1

E_1, H_1

$\Delta H = -\Delta E. < 0$ ($\because \Delta E > 0$)



very slowly → straight line,

At variable stress : $\int_{-a}^{+a} \cdot \frac{1}{2}\ S \cdot dx \quad d(x)$

$$f_{total} = -\frac{\partial(\Delta G)}{\partial(2a)} \qquad \Delta G = \Delta H + (2a \cdot \gamma) \cdot 2$$

$$\Rightarrow fu = -\frac{\partial H}{\partial(2a)} + 2\gamma = \frac{\partial}{\partial(2a)} \cdot \left(-\frac{1-\nu}{2\mu} S^2 \pi a^2\right)$$

$$= \boxed{\frac{\pi(1-\nu)S^2 a}{2\mu}} \quad \boxed{\text{linear}}$$

Mode I $$\cancel{G} = -\frac{\partial(\Delta H)}{\partial(2a)} = \frac{\pi(1-\nu)S^2a}{2\mu} = \frac{K_I^2}{E'} \quad \text{where} \quad \left(\begin{array}{l} K_I = S\sqrt{\pi a} \\ E' = \frac{E}{1-\nu} \end{array} \right)$$

Two loadings $$\underbrace{\sigma_{yy}^{(1)} \text{ and } \sigma_{yy}^{(2)}}_{\text{same mode}} \rightarrow G_{\#} = \frac{(K_I^{(1)} + K_I^{(2)})^2}{E'}$$ (super position).

Two modes $$\sigma_{yy} \qquad G = K_I^2/E' + K_{II}^2/E'$$

→ when superposed, orthogonal modes $\rightarrow 0$ $(\vec{K_1}\cdot\vec{K_2} = 0)$.

J-integral.

$G \geq G_c$ is crack growth

$K_I \geq K_{Ic}$

$\boxed{\text{Energy Release Rate}}$.

J-integral $\Rightarrow$ $$J_i = \int_S (w \cdot n_i - T_j\, u_{j,i})\, d\vec{s}$$

path : $dx\,\hat{i} + dy\,\hat{j}$

$\vec{ds}$: $(dy\,\hat{i} - dx\,\hat{j})\cdot ds$

$\vec{n_i}$: $\hat{i}$

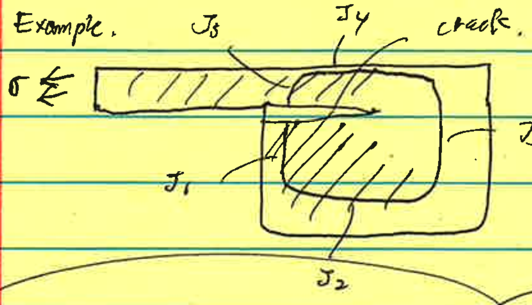
① $w \cdot \vec{n_i} \cdot \vec{ds} = w \cdot dy\, ds$

② $T_j u_{j,i} = \underset{\sim}{T} \frac{d\cancel{u}}{dx} dS$

$= \underset{\sim}{T} \frac{d\underset{\sim}{u}}{dx} dS$

$$\Rightarrow \boxed{J_x = \int_p \cdot w\, dy - \left(T_x \frac{\partial u_x}{\partial x} + T_y \frac{\partial u_y}{\partial \cancel{x}} \right) ds}$$

Example.



$J_1 = \cancel{w}^0, \cancel{\underset{\sim}{T}}^0$

$J_2 = \cancel{dy}^0, \cancel{\underset{\sim}{T}}^0$

$J_3 = \cancel{w}^0{}_{x\to\infty}, \cancel{\underset{\sim}{T}}^0$

$J_4 = w\,\cancel{dy}^0 + \textcircled{\underset{\sim}{T}}$

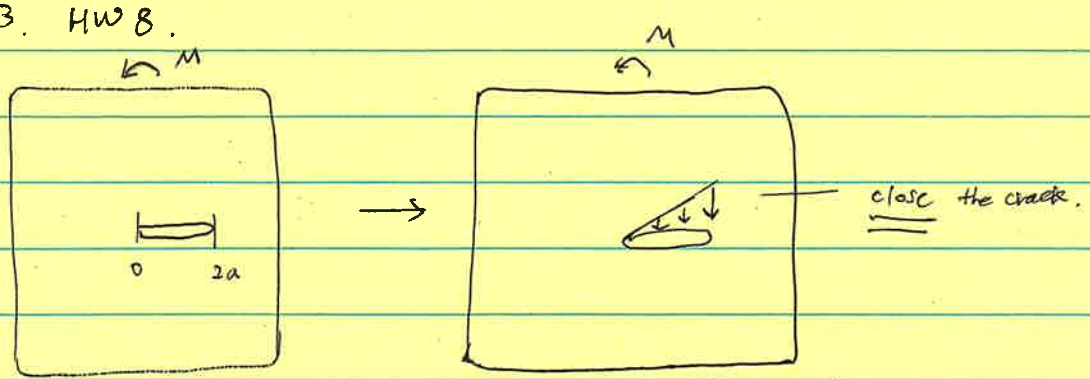
$J_5 = \int w\,dy = \int_b^b \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\,dy$

$$\frac{K_{II}^2}{E'}$$
$$\parallel$$

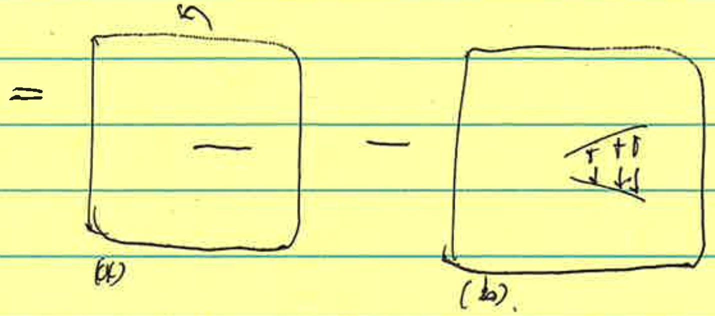
$$-\int_0^b T_x \frac{\partial u_x}{\partial x} dy = -\int_0^b \underset{\sim}{-}\sigma_{xx}\varepsilon_{xx}\,dy = +\sigma^2 b/E'$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \Rightarrow \int_0^b w\,dy = \frac{\sigma^2}{2E'}b + \frac{\sigma^2 b}{E'} = \frac{3\sigma^2 b}{2E'}$$

Pb3. HW8.



$K_I = 0$ $\qquad$ $K_I = 0$ (prevent from crack!)



$K_I^{(a)} = K_I^{(b)}$ $\quad$ $(\because \Sigma K_I = 0)$.

$$\Rightarrow K_I^{(2)} = \frac{E'}{2\underset{\downarrow \; S\sqrt{\pi a} \; (\text{uniaxial})}{K_I^{(1)}}} \cdot \int_{+0}^{+2a} T_i^{(2)}(x) \underset{\downarrow \; S/E' \sqrt{\frac{x}{2a-x}}}{\frac{\partial u_i^{(1)}}{\partial (2a)}} \, \underset{\to dx}{dP} \quad \text{— right side.}$$

$$= \frac{1}{\sqrt{\pi a}} \int_0^{2a} \underset{\overset{\parallel}{\frac{M,x}{I_z}}}{t(x)} \sqrt{\frac{x}{2a-x}} \, dx$$

$$\Rightarrow \boxed{K_I^{(2)} = \frac{3Ma}{2I_z}\sqrt{\pi a}} \quad \text{(right side)}$$

Left side? of crack.

$\frac{Mx}{I_z} \rightarrow \frac{M(2a-x)}{I_z}$

shift!

$$\boxed{K_I' = \frac{1}{2} K_I^{(2)}}$$

Problem Session

05/23/2011

③ $\Delta \varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}}{2\mu} \cdot \frac{1}{2} \{ s_{xx}(t) + s_{xx}(t+\Delta t) \}$ // $\frac{\Delta \bar{\sigma}}{3K} = 0$ ($\because K \to \infty$)

② $\Delta \varepsilon_{xx}^{el} = \Delta s_{xx}^{el} / 2\mu$ $\because$ $\Delta \varepsilon_{xx}^{el} = \Delta \bar{\varepsilon} + \Delta e_{xx}^{el}$ $\left(K = \frac{E}{(1-2\nu)3} \right)$

① $\Delta \bar{\sigma} = \bar{\sigma}(t+\Delta t) - \bar{\sigma}(t)$

$s_{xx}(t) = \sigma_{xx}(t+\Delta t) - \bar{\sigma}_{xx}(t+\Delta t)$

$s_{xy}(t) = \sigma_{xy}(t+\Delta t)$

Equations : ① $\varepsilon_{xx}(t) + \boxed{\Delta \varepsilon_{xx}^{el}} + \Delta \varepsilon_{xx}^{pl} = \varepsilon_{xx}(t+\Delta t)$

$\underline{\varepsilon_{xy}(t) + \boxed{\Delta \varepsilon_{xy}^{el}} + \Delta \varepsilon_{xy}^{pl} = \varepsilon_{xy}(t+\Delta t)}$:

② $J_2 = k^2$

$\frac{1}{2} (s_{xx}^2 + s_{yy}^2 + s_{zz}^2) + s_{xy}^2 = k^2$ (at $t = t+\Delta t$ / $t = t+\Delta t$)

or $\underline{\boxed{\sigma_{xx}^2/3} + \boxed{\sigma_{xy}^2} = k^2}$

$\boxed{\tilde{\lambda}/2\mu}$ are unknown,

fsolve [[trial]

current stress state. ($\hat{\lambda} = 0$)

from the yield point.

Problem Session.   05/17/2024.

· V-M : $J_2 = k^2$, $k^2 = \sigma_Y^2/3$,

$J_2 = \frac{1}{2} s_{ij} s_{ij}$, EPP → $s_{ij}\dot{s}_{ij}$ (∵ $\dot{J}_2 = 0$).

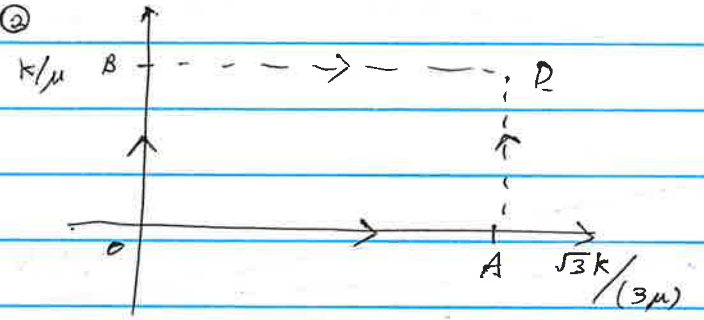
Plastic strain rate : $\boxed{\dot{\varepsilon}_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu} \cdot s_{ij} = \frac{\dot{w}}{2k^2} \cdot S_{ij}}$

· Assumptions : No strain hardening.
ν = 0.5 (incompressible).
V-M criteria.

· σ_{xx}, σ_{xy}

① $J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$ $E = 2\mu(1+\nu) = 3\mu$.

②



(1) Along $\vec{OB}$: Elastic → $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$

$\sigma_{xy} = 2\mu\,\varepsilon_{xy}$ ($\varepsilon_{xy} \in [0, k]$)

(2) Along $\vec{BD}$ ≑ $J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$ (plastic - on the ellipse).

$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{w}}{2k^2} s_{ij}$ & $\underline{\text{no volume change}}$: $\boxed{\dot{w} = S_{ij}\dot{e}_{ij}}$

⇒ ε_{xy} is constant → e_{xy} = constant. → $\dot{e}_{xy} = 0$ ⇒ $\dot{w} = s_{xx}\dot{e}_{xx}$

$S_{xx} = \frac{2}{3}\sigma_{xx}$ · (deviatoric) → $\dot{w} = \frac{2}{3}\sigma_{xx}\cdot\dot{e}_{xx}$ $\boxed{= \sigma_{xx}\dot{\varepsilon}_{xx} \quad \text{(no volumetric)}}$.

⇒ $\dot{\varepsilon}_{xx}^{pl} = \frac{\dot{w}}{2k^2}\cdot S_{xx} = \frac{\dot{w}}{3k^2}\cdot\sigma_{xx}$ ①

Note: $\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{xx}^{el} + \dot{\varepsilon}_{xx}^{pl} = \dot{\sigma}_{xx}/E + \frac{\dot{w}}{3k^2}\sigma_{xx}$ (∵ ①

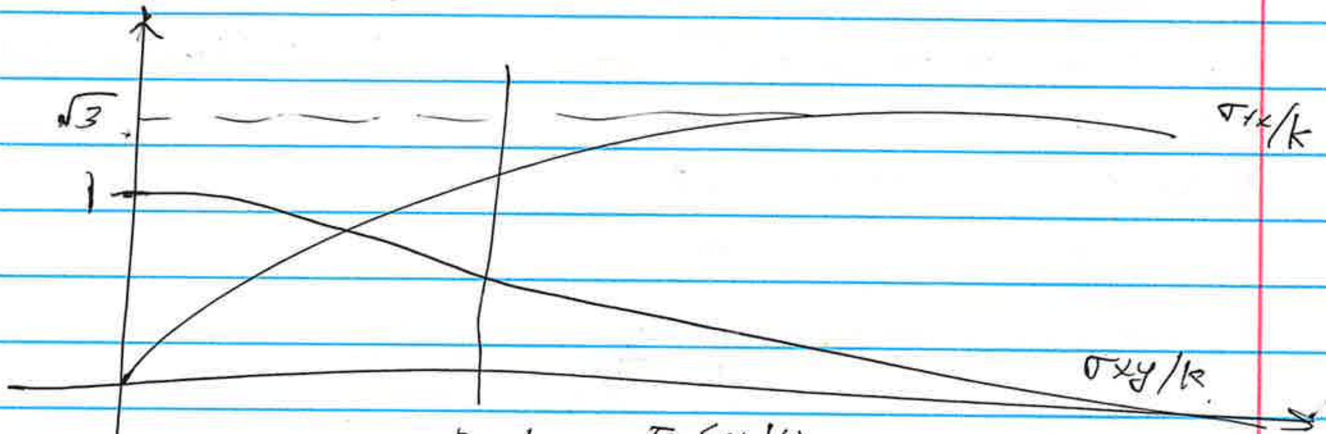
$= \dot{\sigma}_{xx}/E + \frac{\dot{\varepsilon}_{xx}}{3k^2}(\sigma_{xx})^2$

⇒ $\dot{\varepsilon}_{xx}\left(1 - \frac{\sigma_{xx}^2}{3k^2}\right) = \dot{\sigma}_{xx}/E$ → Solve ODE.

simplify, $$\frac{\sigma_{xx}(t)}{\sqrt{3}k} = \tanh\left(\frac{E\ \varepsilon_{xx}(t)}{\sqrt{3}\ k}\right),$$ ✓

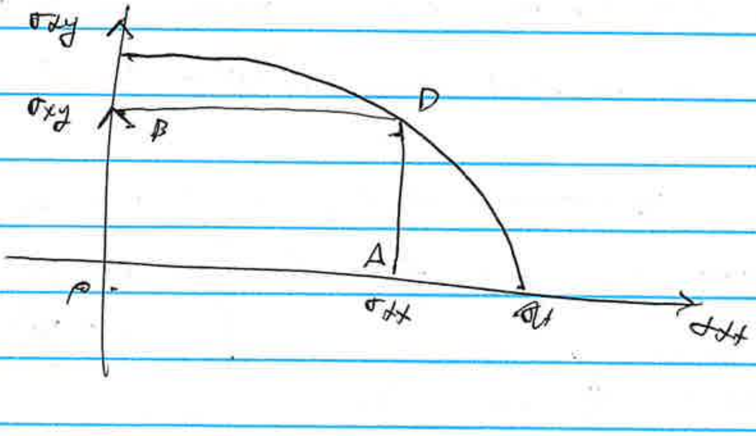
↓ J_2

↘ σ_{xy}



$$D = 1 = \frac{E\ \varepsilon_{xx}(t)}{\sqrt{3}k} \quad (\text{yield})$$

At yield, $\varepsilon_{xx}(t) = \sqrt{3}k/E$. ⇒



Q) What if $\nu \neq 0.5$. $(\nu < 0.5)$ not incompressible.

$\downarrow$

No analytical sol $\rightarrow$ <u>Numerical methods.</u>

Plane-strain $\Rightarrow$ $\underset{\sim}{\sigma_{xx}}$, $\underset{\sim}{\dot{\varepsilon}_{xx}}$ $(\sigma_{yy} = 0)$

$$\sigma_{yy} = \nu \, \sigma_{xx}$$

$$\Rightarrow \quad J_2 = k^2 = \frac{1}{3}\sigma_Y^2 = \frac{1}{2}\left(s_{xx}^2 + s_{yy}^2 + s_{zz}^2\right)$$

$$\Rightarrow \quad \underline{\sigma_{xx} = \frac{\sigma_Y}{\sqrt{1-\nu+\nu^2}}} \;/\!/ \quad > \sigma_Y \qquad (\nu < 0.5).$$

Finite time steps.

At t, $\sigma_{xx}(t)$, $\sigma_{yy}(t)$. $\rightarrow$ $\varepsilon_{xx}(t)$, $\varepsilon_{yy}(t)$

$\downarrow$?

(3 unknowns : $\varepsilon_{xx}(t+\Delta t)$, $\varepsilon_{yy}(t+\Delta t)$, $\frac{\tilde{\lambda}}{2\mu}$).

$$\begin{cases} \Delta\bar{\sigma} = \bar{\sigma}(t+\Delta t) - \bar{\sigma}(t). \\ \Delta s_{xx} = s_{xx}(t+\Delta t) - s_{xx}(t) \\ \Delta s_{yy} = s_{yy}(t+\Delta t) - s_{yy}(t). \end{cases}$$

Total strains. (elastic).

$$\Delta \varepsilon_{xx} = \Delta\bar{\varepsilon} + \Delta e_{xx} = \Delta\bar{\sigma}/3K + \frac{\Delta s_{xx}}{2\mu}.$$

$$\Delta \varepsilon_{yy} = \quad ''$$

Plastic strain.

$$\Delta \varepsilon_{xx}^{pl} = \frac{\tilde{\lambda}}{2\mu} \cdot s_{xx} \cdot \Delta t \qquad \because \; \dot{\varepsilon}_{xx}^{pl} = \tilde{\lambda}/2\mu \cdot s_{xx}.$$

$$\sim \frac{\tilde{\lambda}}{2\mu}\,\Delta t\left(\frac{1}{2}\left(s_{xx}(t+\Delta t) + s_{xx}(t)\right)\right)$$

$$\left.\begin{array}{l}
\varepsilon_{xx}(t) + \Delta\varepsilon_{xx}^{el} + \Delta\varepsilon_{xx}^{pl} - \varepsilon_{xx}(t+\Delta t) = 0 \\
\varepsilon_{zz}(t) + \underline{\qquad} \quad '' \qquad - \varepsilon_{zz} \; '' \quad ' \\
\frac{1}{2}\left(s_{xx}(t+\Delta t)^2 + s_{yy}(t+\Delta t)^2 + s_{zz}(t+\Delta t)^2 \right) = k^2
\end{array}\right\}$$

$\hookrightarrow$ Solve

$$\boxed{\dot{\varepsilon}^{pl}_{ij} = \frac{\dot{W}}{2k^2} S_{ij}}$$

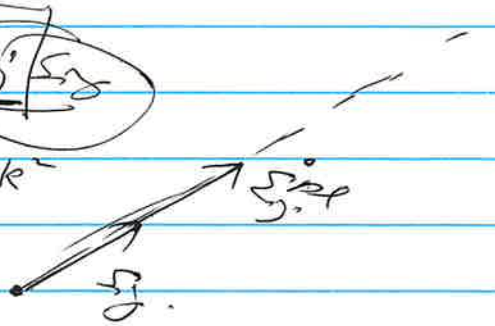
$$S_{ij}\,\dot{\varepsilon}^{pl}_{ij} = \lambda\, S_{ij} S_{ij}$$

$2k^2$

deviatoric $\dot{W} = S_{ij}\,\dot{e}_{ij}$

$S_{ij}\,\dot{e}^{el}_{ij}$

$\dot{e}^{el}_{ij} \to \frac{\dot{S}_{ij}}{2\mu}$



~~$S_{ij}\,\dot{\varepsilon}^{pl}_{ij} = \frac{\dot{W}}{2k^2} S_{ij} S_{ij}$~~

$$J_2 = \frac{1}{2} S_{ij} S_{ij}$$

$$\boxed{\dot{\varepsilon}^{pl}_{ij} = \tilde{\lambda}\, S_{ij}}$$

$$\frac{1}{2} S_{ij}\,\dot{\varepsilon}^{pl}_{ij} = \hat{\lambda}\, \frac{1}{2} S_{ij} S_{ij} = \hat{\lambda} \cdot k^2$$

$$\hat{\lambda} = \frac{\dot{W}^{pl}}{2k^2} = \frac{\dot{W}}{2k^2}$$

← deviatoric pl + el

$$\dot{W} = S_{ij}\left(\dot{\varepsilon}^{pl}_{ij} + \dot{e}^{el}_{ij}\right)$$

$$= S_{ij}\left(\dot{\varepsilon}^{pl}_{ij} + \frac{\dot{S}_{ij}}{2\mu}\right)$$

[illegible] el

$$\dot{J}_2 = S_{ij}\,\dot{S}_{ij} = 0$$