Differential & Variational Equations 01/09/2025.

$\boxed{\text{Elliptic PDE}} \to \boxed{\text{Variational}} \to \left\{ \begin{array}{l} ① \text{ Variational numerical} \\ ② \text{ finite element space} \end{array} \right\} \equiv$ FEM.

- Differential Equation

$$-(k(x)\, u'(x))' + b(x)\, u'(x) + c(x)\, u(x) = f(x) \quad (x \in \underbrace{(0, L)}_{\Omega}) \quad \text{——— ①}$$

→ second order PDE.

→ u must be smooth, computable, satisfy the eq. for $x \in \Omega$.

$(u : \Omega \to \mathbb{R}.)$

- Boundary Conditions

$\Omega = (0, L)$, $\partial\Omega = \{0, L\}$, $\underbrace{\bar{\Omega} = \Omega \cup \partial\Omega = [0, L]}_{\text{closure of } \Omega}$

1) Dirichlet : $u(0) = g_0$ $\quad u(L) = g_L$ ——— ②

2) Neumann : $u'(0) = d_0$ $\quad u'(L) = d_L$ ——— ③

- Boundary Value Problem (BVP)

$$-(k(x)\, u'(x))' + b(x)\, u'(x) + c(x)\, u(x) = f(x) \quad (x \in \underbrace{(0, L)}_{\Omega})$$

$$u(0) = g_0$$

$$u'(L) = d_L$$

- Theorem (existence & uniqueness)

Assume k, b, c, f are smooth + bounded and (1) $k(x) \geq k_0 > 0$

(2) $c_0 = \min_x c(x)$, $c_0 \geq 0$

Then, BVP has a unique solution (* works for $\underbrace{(D, D), (D, N)}_{\text{boundary conditions.}}$ but not (N, N))

- Example of variational problem.

Goal : $u^3 + \ln u - 1 = 0$,

Let $R(u,v) = (u^3 + \ln u - 1)v \Leftrightarrow R(u,v) = 0 \quad \forall v \in \mathbb{R}$.

satisfies for $R(u,v)\big|_{v=1} = 0$ and $R(u,v)\big|_{v=2} = 0$ .....

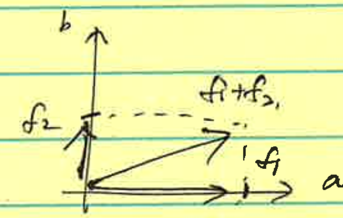
- Vector space.

① closure ② , , , , ⑥.

$x+y \in V$ $\alpha x \in V$

Eg.). $V = \{$ 2nd order polynomial 0 at $x=0\}$ → ax^2+bx

$f_1 = 3x^2$, $f_2 = x \Rightarrow f_1 + f_2 = 3x^2 + x$



- V is set of functions in $\Omega \subset \mathbb{R}^n$ and let $f, g \in V$, $\alpha \in \mathbb{R}$.

① $h(x) = f(x) + g(x)$ defined $\forall x \in \mathbb{R}$.

② $p(x) = f(x) \cdot g(x)$

Note : Identity is $(z(x) = 0)$ where $z(x)$ is a function yields zero.

① Example: $V_2 = \{ f : [a,b] \to \mathbb{R} \mid f(a) = f(b) = 0 \}$. → YES!

② Example : $U_2 = \{1, x, x^2\}$ span $(U_2) = ax^2 + bx + c$. and "Independent!"

Independence is examined using $\sum_i c_i \cdot e_i = 0 \Leftrightarrow c_i = 0$

- $\mathcal{Y}, \mathcal{V}$ and $R : \mathcal{Y} \times \mathcal{V} \to \mathbb{R}$

$R(u, v + \alpha w) = R(u,v) + \alpha R(u,w)$: Linearity.

Variational equation : $R(u,v) = 0 \quad \forall v \in \mathcal{V}$

$\Rightarrow u \in \mathcal{Y}$ is a solution.

Ω

- Variational Formulation.

$$-(k(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x) \quad \forall x \in \Omega$$

Residual : $r(x) = -(ku')' + bu' + cu - f = 0$

$$R(u,v) = (u(0) - g_0)v(0) = 0 \qquad v \in V \qquad \text{(Boundary)}$$
$$R(u,v) = r(x_1)v(x_1) = 0 \qquad v \in V \qquad \text{(PDE)}$$

Ex) $-u'' = f \qquad (x \in (0,L) = \Omega)$

$u(0) = g_0 \quad u'(L) = d_L$

① $r = -u'' - f$

② $\underbrace{\int_\Omega r(x)v(x)\,dx}_{= R_1} = 0 \qquad \forall v \in V = \{f : (0,L) \to \mathbb{R} \text{ smooth}\}$

③ $R_1 = \int_0^L (-u'' - f)v\,dx = \int -u''v - \int fv = \underbrace{\int u'v' - [u'v] - \int fv}_{= R_2(u,v) = 0}$.

$\Rightarrow$ ④ $R_2 = -\underbrace{u'(L)}_{= d_L \ (B.C.)} v(L) + u'(0)v(0) = R_3(u,v)$.

⑤ Enforce $v(0) = 0 \Rightarrow \boxed{R(u,v) = \int u'v' - d_L v(L) - \int fv}$ "PDE + BCs"

$V = \{v : [0,L] \to \mathbb{R} \text{ smooth} \quad v(0) = 0\}$.

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Variational Equations — Euler-Lagrange.

① $r(x) = -(ku')' + bu' + cu - f = 0$: residual

② $v \in C^\infty(\Omega) \Rightarrow \int_0^L r(x)\, v(x) = 0$: multiply

③ $\int -(ku')'v + \int bu'v + \int cuv - \int fv = 0.$

$\Rightarrow -\left. ku'v \right|_0^L + \int \left(ku'v' + bu'v + cuv - fv \right) dv = 0$: integrate

④ $u(0) = g_0$, $u'(L) = d_L$, impose $\forall v \in V^2 = \{ v(0) = 0 \}$

$- k\, d_L \cdot v(L) + \int_0^L (ku'v' + bu'v + cuv - fv)\, dv = 0$

- we used $u'(L) = g(L) \rightarrow$ Natural Boundary Condition (NBC)
- we didn't use $u(0) = g(0) \rightarrow$ Essential Boundary Condition. (EBC)

Ex) $L = \pi/2$, $g_0 = 1$, $d_L = -1$, $-u'' = f$

$u(x) = \cos x$, set $v(x) = \sin x$

$\int_0^{\pi/2} (//) - \int (///) = 0$

Let $u_1' = u' + 1 \Rightarrow u_1 = u + 1$, $u_1' = u'$, $\Rightarrow$ satisfies variational equation.

Let $u_2 = u + x \Rightarrow u_2(0) = g_0$, $u_2' \neq u' \Rightarrow$ doesn't satisfy variational equation.

$u_2'' = u \rightarrow$ satisfies PDE.

- Nitsche's Method

$$\left.\begin{aligned} &(g - u(0))\, v'(0) = 0 \\ &\mu (u(0) - g_0) \cdot v(0) = 0 \end{aligned}\right\} \Rightarrow \text{Add to } R(u, v)$$

• Euler – Lagrange Equation

$$R(u,v) = \int u'v' - d_L v(L) - \int f v = 0 \qquad \text{for} \left(\begin{array}{l} u'' = f \\ u(0) = g_0 \\ u'(L) = d_L \end{array} \right.$$

① Integrate

$$u'v \Big|_0^L - \int u'' v - d_L v(L) - \int f v = 0$$

② Simplify

$$u'(L)\, v(L) - \cancel{u'(0) v(0)}^{\,0} - d_L \cdot v_L - \int_0^L (u'' + f) v \; dx = 0$$

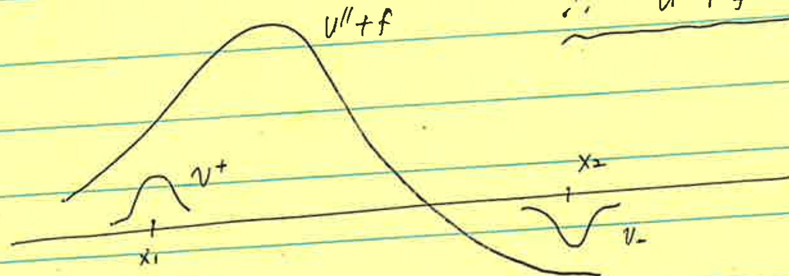
$$\Rightarrow \underline{\left(u'(L) - d_L \right) v(L) - \int_0^L (u'' + f)\, v = 0}.$$

choose $v(L) = 0$. ?

③ $v^+ > 0$ in $u'' + f \leq 0$ in neigh x_1

$v^- < 0$ in $u'' + f \geq 0$ in " x_2

$\therefore$ $u'' + f = 0$ — trivial!



④ $(u'(L) - d_L)\, v(L) = 0$

$\therefore$ $u'(L) - d_L = 0$ must be satisfied.

$$EL(u,x) = \left. \begin{array}{c} u'' + f \\ u' - d_L \end{array} \right\}$$

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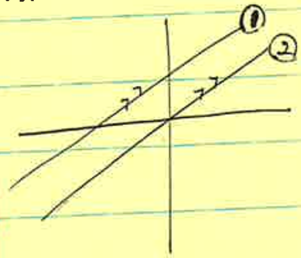
Variational Numerical Method



① not a vector space → affine ~~space~~ sub. space.

② Vector space

Ex) $V = \{ w : [a,b] \to \mathbb{R}$ smooth $C^\infty \mid w(a) = w(b) = 0 \}$

$V' = \{ w : [a,b] \to \mathbb{R} \quad C^\infty \mid w(a) = 1, w(b) = 2 \}$

$\Rightarrow w_2, w_1 \in V' \Rightarrow w_2 - w_1 \big|_a = w_2 - w_1 \big|_b \equiv 0 \Rightarrow$

$\therefore$ V' is affine subspace of V.

- **Var. Form.**

$R(u,v) = 0 \iff a(u,v) = \ell(v)$. $\quad u \in \mathcal{S}$

$\downarrow$

$$EL = \begin{cases} u'' + f & (x \in \Omega) \\ u'(x) - d_L & (x = L) \end{cases}$$

Weak Form.

Define $\mathcal{S} = \{ w \mid C^\infty,\ w(0) = g_0 \} \Rightarrow$ Find $u \in \mathcal{S}$, $R(u,v) = 0$.

- **Bilinear Form**

$$\left. \begin{aligned} a(u + \alpha v, w) &= a(u,w) + \alpha a(v,w) \\ a(u, w + \alpha z) &= a(u,w) + \alpha a(u,z) \end{aligned} \right)$$

If $a(u,v) = a(v,u) \rightarrow$ "Symmetric Bilinear Form."

- $R(u,v) = a(u,v) - \ell(v)$. $\Rightarrow \boxed{a(u,v) = \ell(w)}$

1) $u = 0$, $R(u,v) = -\ell(v) \neq 0$

Not linear in u → Affine subspace.

2) Linear in v → Vector space.

- Linear Variational Equation.

$a(u,v) = \ell(v)$

To be linear f, $f(0)=0$. must be satisfied.

$R(u,v) = a(u,v) - \ell(v)$

$R(u,0) = \cancel{a(u,0)}^{0} - \cancel{\ell(0)}^{0} = 0$ [Linear in u]

$R(0,v) = a(0,v) - \ell(v) \neq 0$ [Affine in v]

But, $a(u,v)$ is linear in u and v

$\Rightarrow$ $\boxed{a(u,v) \text{ is bilinear}, \ \ell(v) \text{ is linear}}$

- Basis. $V = \{ e_1 \sim e_n \}$

$v = c_1 e_1 + \cdots + c_n e_n$ where c_i : components and e_i are basis.

- Classical Galerkin Method.

Ex). $-u'' + bu' + u = 0 \quad 0<x<1, \quad u(0)=3, \quad u'(1)=0$

$$\left(\begin{aligned} a(u,v) &= \int_0^1 u'v' + bu'v + uv \, dx \\ \ell(v) &= \int_0^1 0 v \cdot dx = 0 \\ V &= \{ v : [0,1] \to \mathbb{R} \mid v \in C^n \wedge v(0) = 0 \} \end{aligned} \right)$$

① $W_h = \text{span}\{1, x, \cdots, x^p\} = \mathbb{P}_p$.

② $S_h = \{ w_h \in W_h \mid \underbrace{w_h(0) = 3}_{} \}$ $\Rightarrow$ Affine subspace! $\Rightarrow w_h = 3 + c_1 x + \cdots + c_p x^p$

↳ Essential Boundary Condition.

③ V_h = direction of S_h

$= \{ w_h \in W_h \mid w_h(0) = 0 \} \Leftrightarrow w_h = c_1 x + c_2 x^2 + \cdots + c_p x^p$

$\Rightarrow$ $\boxed{\text{Find } u_h \in S_h \ \text{ s.t. } \ a(u_h, v_h) = \ell(v_h) \ \text{ for } \forall v_h \in V_h}$

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- Classical Galerkin Method.

$R(u,v) = \underbrace{a(u,v)}_{\text{bilinear}} - \underbrace{\ell(v)}_{\text{linear}} \overset{!}{=} 0$ (with natural BC)

$\Rightarrow \underline{a(u,v) = \ell(v)}$ "

$u \in S_h$ ↓ affine subspace ↓ by essential BC.

$v \in V_h$ ↓ vector space ↓ w/ essential BC

- S_h : Trial Space (affine subspace) → essential B.C
  V_h : Test Space. (vector space).

- Question : $u(0) = 3$, $u'(1) = 0$ with PDE. (example).

$W_h = \text{span}\{1, x, \cdots, x^p\} \Rightarrow w_h \in W_h$: $\boxed{w_h = c_0 + c_1 x^1 + \cdots + c_p x^p}$

$S_h = \{w_h \in W_h \mid w_h(0) = 3\}$: $\boxed{u_h = 3 + c_1 x^1 + \cdots + c_p x^p}$ DoF : p. (p unknowns)

$V_h = \{\text{direction of } S_h\}$
$= \{w_h \in W_h \mid w_h(0) = 0\}$: $\boxed{v_h = 0 + c_1 x^1 + \cdots + c_p x^p}$ DoF : p. (p equations)

$R_h = R$.

$R_h(u_h, v_h) = 0 \Rightarrow a_h(u_h, v_h) = \ell(v_h)$

$$\left.\begin{aligned} \Rightarrow\ & a_h(3 + c_1 x + \cdots + c_p x^p, x^1) = \ell(x^1) \\ & \vdots \\ & a_h(3 + c_1 x + \cdots + c_p x^p, x^p) = \ell(x^p). \end{aligned}\right\} \begin{pmatrix} p \text{ unknowns} \\ p \text{ equations} \end{pmatrix} \rightarrow \text{solve!}$$

$$\underset{\sim}{A}\, \overset{?}{c} = \underline{\ell}.$$

λ.
E.

- Consistency.

check if $a_h(u, v_h) = l_h(v_h) \Rightarrow a(u, v_h) = l(v_h)$. $\forall v_h \in V_h$

we know, $a(u,v) = l(v)$. for $\forall v \in V\{v \mid v(0)=0\}$

If $v_h \subset V$, $\rightarrow$ Consistent!

Ex) if $V = \{\text{span}\{1, x, \cdots, x^{p-1}\}\}$,

$R(u, v_h) = \int \sim + u'(0)\, v_h(0)$ $\rightarrow$ not necessarily zero.

$\Rightarrow$ Not consistent.

- Discrete Variational Problem.

$(m \in \mathbb{N})$ $(m!)$

① W_h : Chose a basis $\{N_1, \cdots, N_m\}$

② $u_h(x) = \sum_{b=1}^{m} u_b \cdot N_b(x)$

$v_h(x) = \sum_{a=1}^{m} v_a N_a(x)$

$\{N_1, \cdots, N_n\}$ is a basis of V_h. $(m > n)$

③ Assume $n \leq m$.

$\underbrace{N_1 \sim N_n}_{\text{basis of } V_h}, N_{n+1}, \cdots, N_m$. $\Rightarrow N_a = 0$ for $a > n$.

$\Rightarrow v_h(x) = \sum_{a=1}^{n} V_a N_a(x)$.

④ $a_h(u_h, N_a) = l_a(N_a)$ $(a = 1, \cdots, n)$ $\rightarrow$ n equations.

⑤ Chose any $\bar{u}_h \in S_h \Rightarrow \bar{u}_h(x) = \underbrace{\bar{u}_1 N_1 + \cdots + \bar{u}_n N_n}_{\in V_h \ (1)} + \underbrace{\bar{u}_{n+1} N_{n+1} + \cdots + \bar{u}_m N_m}_{\notin V_h \ (2)}$.

$\{u_a = \bar{u}_a \quad (a = n+1, \cdots, m)$ $\rightarrow$ Boundary condition.

$\because$ (2) part must be identical everytime since we only have 'n' equations.

$\Rightarrow$ u_a $a = n+1 \sim m$. are constants.

⑥ $l(N_a) = a(u_h, N_a) = a\left(\sum_{b=1}^{m} u_b \cdot N_b, N_a\right) = \sum_{b=1}^{m} u_b\, a(N_b, N_a)$

(for $a = 1 \sim n$)

⑦ $F_a = l(N_a)$ $\quad K_{ab} = a(N_b, N_a)$ $\quad a = 1 \sim n \ , \ b = 1 \sim m$

$F_a = \bar{\mu}_a$ $\quad K_{ab} = \delta_{ab}$ $\quad a = n+1 \sim m \quad b = 1 \sim m$

$$\Rightarrow \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1m} \\ \vdots & & & \vdots \\ k_{a1} & & & k_{am} \\ \vdots & & & \\ k_{m1} & \cdots & \cdots & k_{mm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}.$$

stiffness matrix $\qquad$ load vector.

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- Consistency : Cont. $\leftrightarrow$ Disc.

$$\begin{array}{c} \\ n+1 \\ \\ \end{array}\left[\begin{array}{c|c} \multicolumn{2}{c}{K_{ab} = a(N_b, N_a)} \\ \hline 0 & I \end{array}\right] \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ u_{n+1} \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \ell(N_1) \\ \vdots \\ \ell(N_n) \\ \bar{u}_{n+1} \\ \vdots \\ \bar{u}_m \end{bmatrix}$$

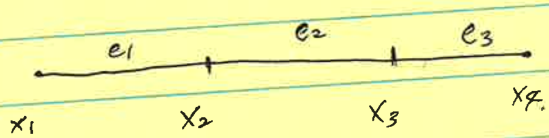
$$K \qquad U = F.$$

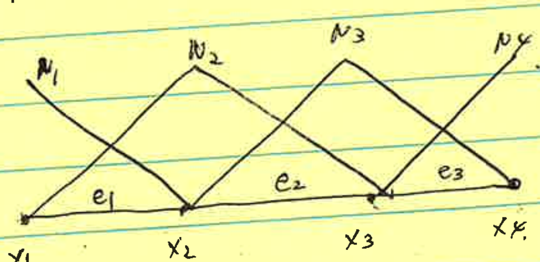
- Basis

$\eta = \{1, 2, \cdots, m\}$

$\eta_a \in \eta$. "active indices" $\rightarrow V_h = \mathrm{span}\left(\bigcup_{b \in \eta_a} \{N_b\} \right)$

$\eta_g = \eta \setminus \eta_a$ "constrained indices"

- 



$\rightarrow$ Basis functions.

$\sum N_i = 1$

$N_a(x_b) = \delta_{ab}$ $\left(\text{Ex)}\ N_3(x_3) = 1,\ N_3(x_2) = 0 \right)$

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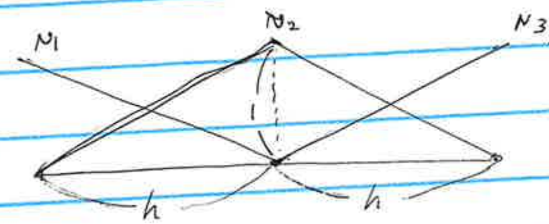
Variational Numerical Method.

- Example



$$a.(N_b, N_a) = K_{ab} = \int_0^1 N_b' N_a' \, dx$$
$$\ell\,(N_a) = F_a = \int_0^1 1 \cdot N_a \, dx.$$

$\ell(N_1) = h/2$, $\ell(N_2) = h$, $\ell(N_3) = h/2$.

$a.(N_1, N_1) = k_{11} = (-1/h)^2 \cdot h. = 1/h.$

$a.(N_1, N_2) = k_{21} = (-1/h.)(+1/h).h = -1/h.$

$a.(N_2, N_2) = k_{22} = (-1/h.)^2 \cdot h + (+1/h)^2 \cdot h = 2/h.$

$\begin{cases} \text{If. } N_1 \text{ is constrained, not active} \\ k_{12} = k_{13} = 0, \quad k_{11} = 1. \end{cases}$

- Consistency

u solves BVP $\xrightarrow[\text{consistency}]{}$ $R_h(u, v_h) = 0$ for $v_h \in V_h$.

$\Rightarrow$ $R(u, v) = 0.$ for $\forall v \in V$

$R_h = R$, $V_h \subset V$ $\Rightarrow$ then, automatic consistency.

V should be smooth, but not smooth 

If). It is consistent using 

$$R_h(u, v_h) = \sum_{i=2}^{n_{el}} u'(x_i) \underline{[[\, v_h(x) \,]]}_{x = x_i} = 0$$
$$= 0 \;.=\; v_h(x_i^+) - v_h(x_i^-).$$

Only continuity gives Consistency.

shape functions.

- Element space. $P^e = \text{span}\{N_1^e, \dots, N_k^e\} = \text{span}\, N^e$.

k: # of D.O.F. : $\{\phi_1^e, \dots, \phi_k^e\}$: DOF

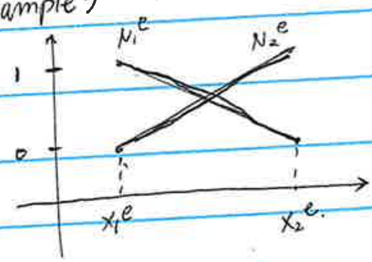
$f^e \in P^e \Rightarrow f^e : K^e \to \mathbb{R}$.

$f^e(x) = \phi_1^e N_1^e(x) + \dots + \phi_k^e N_k^e(x)$.

$e = (K^e, N^e)$

element.

Example.) P_1 element.



Independent,

$\text{span}\{N_1^e, N_2^e\} = P_1(K^e)$

- Node.

$\phi_1^e = f(x_1^e)$ $\quad$ $\phi_2^e = f(x_2^e)$ $\Big\}$ Then, we call them "nodes"

D.O.F = 1.

$f^{(j)}(x)$ is dof at the node.

for some j, any f

- $\underbrace{\{P_1, P_2, \dots, P_K\}}$

$u(\cdot)$

$F(u+) - F(u-)$

$\{K, F\}$.

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- $f \in P^e \Rightarrow f(x) = f(x_1) N_1(x) + \cdots + f(x_{k+1}) N_{k+1}(x)$.

- what is node ?.

Ex) $e = \left([x_1, x_2], \{1, x\} \right) \rightarrow$ Doesn't have nodes.

$f \in P^e \Rightarrow f = \phi_1 + \phi_2 \cdot x$.

If there is $\bar{x} \in [x_1, x_2]$ s.t. $f(\bar{x}) = \phi_1$, $f(\bar{x}) = \phi_2$, there is node.

$\rightarrow$ suppose $[1, 2]$

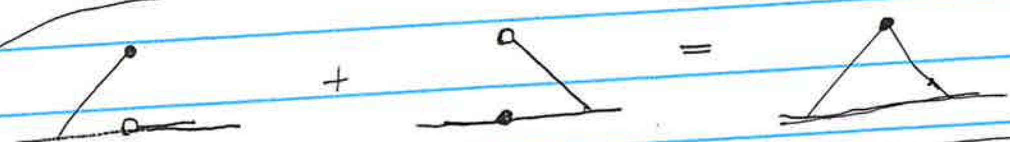
$f(\bar{x}) = \phi_1$, $\phi_2 \neq 0$, no $\bar{x}$ exists.

$f(\bar{x}) = \phi_2 \Rightarrow \bar{x} = \dfrac{\phi_2 - \phi_1}{\phi_1} \rightarrow$ depends on f. $\rightarrow$ no $\bar{x}$.

- $W_h = \mathrm{span}\{N_1, \cdots N_n\}$.

↳ choose basis. functions.

Idea :



- Local to Global Map.

$LG(a, e) = \begin{array}{c} \xrightarrow{\text{elements}} \\ \left[\quad \right] \end{array} \Big\downarrow$ shape funcs.

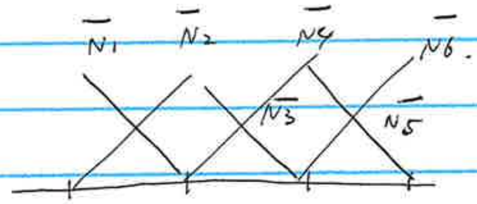
shape func. index, element

Broken Sum

$A = \{1 \sim nn\}$, $N_A : \Omega \rightarrow \mathbb{R}$ is, $N_A(x) = \sum_{(a,e) \mid LG(a,e) = A} N_a^e(x)$

when $x \neq x_i$ (not a vertex). $N_A(x_i) = \lim_{x \to x_i} N_A(x)$.

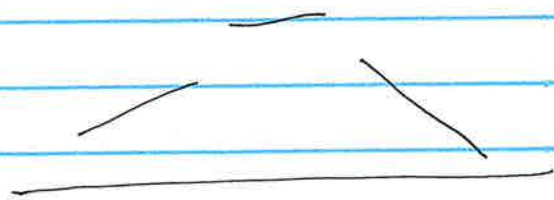
- Example



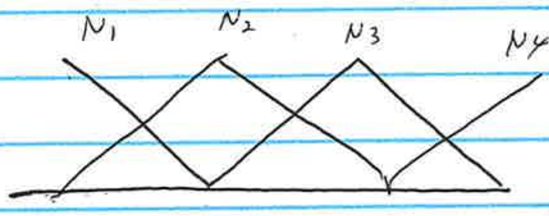
$$LG = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$LG(2,1) = 2. \rightsquigarrow N_2$$

2nd shape   first element.



$: u = 1\bar{N}_1 \dotplus 2\bar{N}_2 \dotplus \cdots \dotplus 0\bar{N}_6.$

- 

$N_1 = \bar{N}_1$

$N_2 = \bar{N}_2 + \bar{N}_3$

$N_3 = \bar{N}_4 + \bar{N}_5$

$N_4 = \bar{N}_6$

elements. $\longrightarrow$

$$\begin{array}{cc} \text{shape} \downarrow & \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \end{array}$$

(2nd of element 2.
1st of " 3.)

(2nd of element 1
1st of " 2)

Ex) $N_A = \sum_{(a,e)\,|\,LG(a,e)=A} N_a^e$

So $N_2 = \sum_{(2,1),(1,2)} N_a^e = N_1^2 + N_2^1 \Rightarrow$ True.

$\Rightarrow K_{ab}^e = a_h^e(N_b^e, N_a^e)$

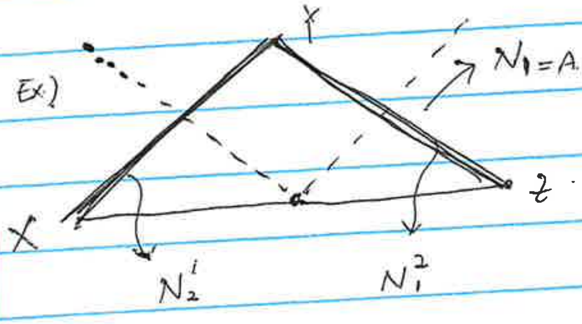
$F_a^e = l_h^e(N_a^e)$

02/04/2025.

- (A) Global Basis Function $= \sum_{(a,e)}$ Local Basis Function.

where $LG(a,e) = A$.

Ex)



$XY : N_2^1 \quad (a,e) = (2,1)$

$YZ : N_1^2 \quad (a,e) = (1,2)$

$XYZ : N_1 \quad A = 1.$

$$LG = \begin{bmatrix} ? & 1 & \cdots \\ 1 & ? & \end{bmatrix} \Rightarrow LG(a,e) = A$$

$$\therefore \; u_h(x) = \sum_{LG(a,e)=A} N_a^e(x)$$

- Element Stiffness Matrix & Load Vector.

Global Stiffness Matrix

(1) $a(u,v) = a(N_B, N_A) = a\left(\sum_{LG(a,e)=B} N_a^e, \sum_{LG(a,e)=A} N_a^e\right) = K_{AB}.$

$= \sum_{LG(a,e)=A} a\left(\sum_{LG(a,e)=B} N_a^e, N_a^e\right) = \sum_{LG(a,e)=A} \sum_{LG(a',e')=B} a\left(N_{a'}^{e'}, N_a^e\right).$

Local Stiffness Matrix for each "element".

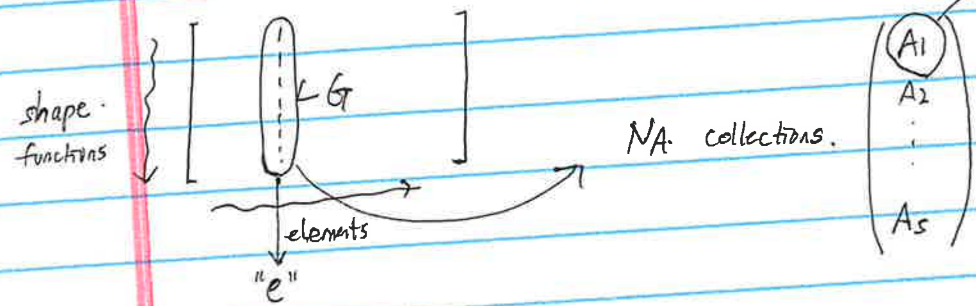
(2) $l(v) = l(N_A) = l\left(\sum_{LG(a,e)=A} N_a^e\right)$

$= \sum_{LG(a,e)=A} l(N_a^e) \rightarrow$ Local load vector for each "element".

K^e and F^e for each element.

- $K^e_{ab} = a^e_h(N^e_b, N^e_a)$, $F^e_a = l^e_h(N^e_a)$.

For element "e". ~~find a(a,b)~~ We know $\underline{\underline{(a,e)}}$ for A_1



Note: If. $l = \sum l^e_h + \underbrace{h_0 v_h(0)}_{} + \underbrace{h_L v_h(L)}_{}$

depends on the elements

- $F_A = \sum_e \sum_{\{a \mid LG(a,e)=A\}} F^e_a$ for $A \in \eta_a$.

$K_{AB} = \sum_e \sum_{\substack{\{a \mid LG(a,e)=A\} \\ \{b \mid LG(b,e)=B\}}} k^e_{ab}$ for $A \in \eta_a$, $B \in \eta$

$\Rightarrow K(LG(a,e), LG(b,e)) \mathrel{+}= K^e_{ab}$.

- Weak Form in 2D, 3D, $\cdots$ 02/11/2025.

$$\boxed{\underbrace{\int_\Omega v\,\mathrm{div}\,w\,d\Omega}_{1} = \underbrace{\int_{\partial\Omega} v w\cdot\vec{n}\,d\Gamma}_{2} - \underbrace{\int_\Omega w\cdot\nabla v\,d\Omega}_{3}} \quad \text{——} \quad (*)$$

Example).

$$a(u,v) = \underbrace{\int_\Omega (k\nabla u)\cdot\nabla v\,d\Omega}_{3} = \underbrace{\int_{\partial\Omega} k\nabla u\cdot\vec{n}\,v\,d\Gamma}_{2} - \underbrace{\int_\Omega \mathrm{div}(k\nabla u)\,v\,d\Omega}_{1}$$

$$l(v) = \int_\Omega f v\,d\Omega + \int_{\partial\Omega_N} H v\,d\Gamma$$

$$\Rightarrow a(u,v) - l(v) = \int_{\partial\Omega_N}(k\nabla u\cdot\vec{n} - H)\,v\,d\Gamma + \int_\Omega\{-\mathrm{div}(k\nabla u) - f\}\,v\,d\Omega \to 0.$$

$$+ \cancel{\int_{\partial\Omega_D} k\nabla u\cdot\vec{n}\,v\,d\Gamma}$$

(use $v=0$ for $\partial\Omega_D$).

$\Rightarrow$ Euler Lagrange,

$$k\nabla u\cdot\vec{n} - H = 0 \quad \text{for } \partial\Omega_N.$$

$$\mathrm{div}(k\nabla u) + f = 0 \quad \text{for } \Omega$$

- Variational Numerical Methods.

a_h, l_h, $\quad a_h : S_h \times V_h \to \mathbb{R}$.

$\quad l_h : V_h \to \mathbb{R}$.

① $a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h$ (on Ω), ② $u_h = g$ (on $\partial\Omega$).

*. For diffusion problem, $u_h, v_h \in C^0$

Ex.) $\Omega = \{ (x_1, x_2) \in \mathbb{R}^2, \quad 0 \le x_1 \le L, \ 0 \le x_2 \le L. \}$

$$\begin{cases} \Delta u = -f/k & (x \in \Omega) \\ u = g & (x \in \partial\Omega). \end{cases}$$

r (dimension) = 2, Base space : { polynomial degree $\le r$. }. $= P_r(\Omega)$.

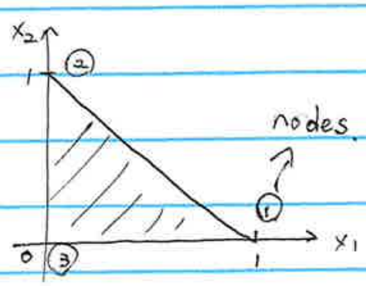
If choose $r=1$, $u = c_1 + c_2 x_1 + c_3 x_2 = 0$ for $(x_1 = 0, x_2)$ $(x_1, x_2 = 0)$

$(x_1 = L, x_2)$ $(x_1, x_2 = L)$.

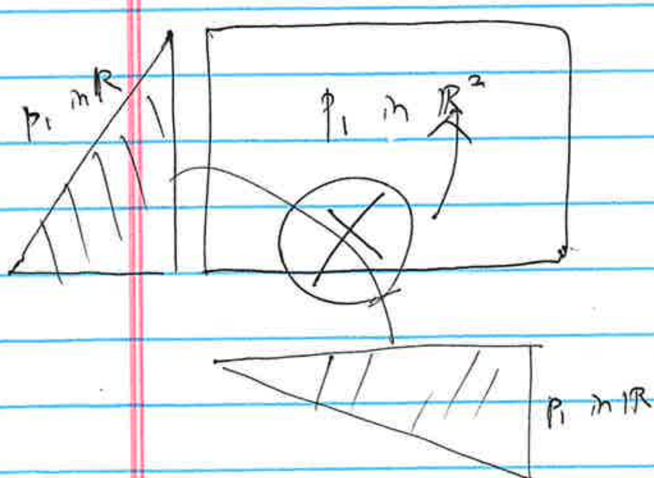
$\Rightarrow u = 0 \rightarrow$ no sense!

$\Rightarrow$ $\boxed{r=2}$ is good choice. $\rightarrow$ $r=4$

- P_1 element in 2D.



$$\begin{cases} N_1 = x_1 \\ N_2 = x_2 \\ N_3 = 1 - x_1 - x_2. \end{cases}$$



$\left.\begin{array}{l} \text{span}(N_a) = P_1(x_1) \\ \text{span}(N_b) = P_1(x_2) \end{array}\right\}$ tensor product.

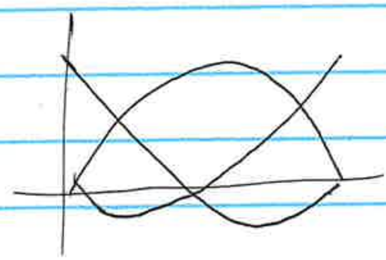
P_2 in $\mathbb{R}^1$ $\xrightarrow{\text{Tensor Product}}$ Q_2 in $\mathbb{R}^2$

$M_1(x) = \sim$

$M_2(x) = \sim$

$M_3(x) = \sim$

$$\left.\begin{array}{l} N_1 = M_1 \cdot M_1 \\ \vdots \\ N_9 = M_3 \cdot M_3 \end{array}\right\},$$



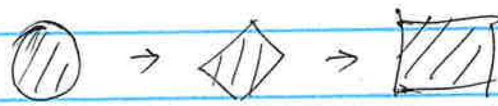
- Element Domain. (Includes Boundaries! ?).

  $B_n = \{ (x_1 \sim x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1 \}$ → unit ball.

  $B_n \to K \in \mathbb{R}^m$ is bijective.

  ⇒ Take unit ball and "warp" it to □

  ex)



- Mesh

→ (thick line) : intercepts mesh.

(thin line) ; out of intercept.

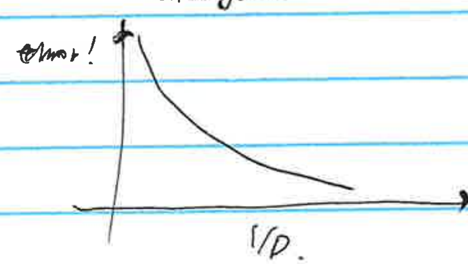
Diameter of element = $\underline{\max(|x-y|)}$ for $(x, y \in E)$.

$\|$

D

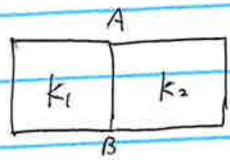
↓

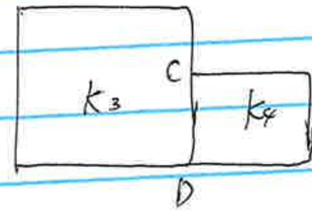
convergence test



02/13/2025

• Conforming Mesh.





$K_1 \cap K_2 = \overline{AB}$

→ Conforming

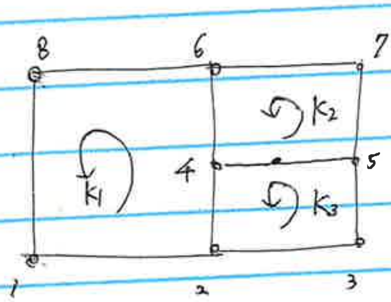
$K_3 \cap K_4 = CD$ → This is not edge.

→ non-conforming.

paper 1

paper 2.

3 ?

• 

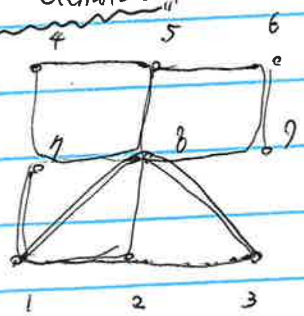
$$LV = \begin{array}{c} \overrightarrow{\text{elements}} \\ \begin{array}{ccc} K_1 & K_2 & K_3 \end{array} \\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 6 & 7 & 5 \\ 8 & 6 & 4 \end{pmatrix} \end{array}$$

Λ

• Convention is counter-clockwise (right hand rule).

• This is non-conformal mesh.

Q_1 - element!

• 

$$LV = \begin{pmatrix} 1 & 2 & 7 & 8 \\ 2 & 3 & 8 & 9 \\ 8 & 9 & 5 & 6 \\ 7 & 8 & 4 & 5 \end{pmatrix}$$ Basis Functions.

$\rho(z)$.

Reynolds Transp. Thm

$-\rho(x)\, a(x)\, \Delta t$

$-a\cdot(x)\,\rho(x,t)$

$= \partial\rho/\partial t$.

$$N_8 = N_3^1 + N_4^2 + N_2^3 + N_1^4$$

$$LG = \begin{pmatrix} 1 & 2 & & 8 \\ 2 & & 8 & \\ 8 & & & \\ & 8 & & \end{pmatrix}$$

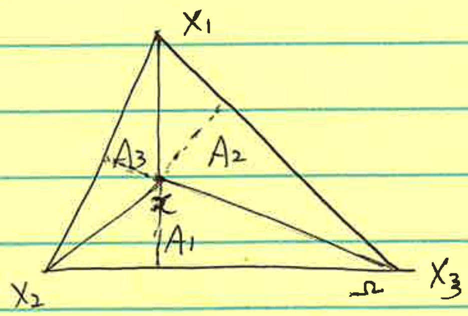
4 D.O.F = LV

If it's good & conformal mesh.

+ Continuous.

02/18/2025.

- Barycentric Coordinates.



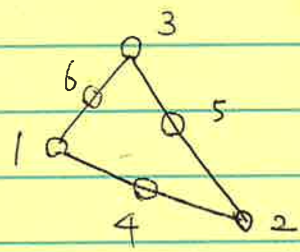
$\lambda_j = A_j / A$

↪ Defines a map. (from basis)

if $\lambda > 0$: inwards $\equiv A_j \cap A \neq \phi$

$\lambda < 0$: outwards $\equiv A_j \cap A = \phi$.

- P_2 of Δ $\quad(\Delta, N)$,



$N_1 = \lambda_1$ $\quad$ ($=1$ only at ①)

($= 1/2$ at ④, ⑥)

↪ we want zero

$$\Rightarrow \begin{cases} N_1 = 2 \cdot \lambda_1 (\lambda_1 - 1/2) = \lambda_1 (2\lambda_1 - 1) \\ N_2 = 2\lambda_2 (\lambda_2 - 1/2) \\ N_3 = 2\lambda_3 (\lambda_3 - 1/2) \end{cases}$$

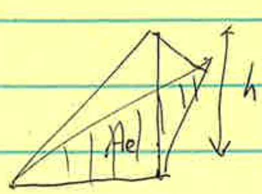
$N_4 = 4 \lambda_1 \lambda_2$, $N_5 = 4 \lambda_2 \lambda_3$, $N_6 = 4 \lambda_1 \lambda_3$. $\rightarrow$ Using Bary. Coord.

- Example.

$$\int_\Omega (k \nabla u) \nabla v \, d\Omega = \int_\Omega f v \, d\Omega + \cancel{\int_{\partial\Omega_N} H v \, d\Gamma}^{\rightarrow 0} \quad , (H = 0) \quad \begin{pmatrix} \partial\Omega = \partial\Omega_D \\ \partial\Omega_N = \phi \end{pmatrix}.$$

$$K^e_{ab} = \int_{\Omega_e} (k \nabla N_b) \nabla N_a \, d\Omega \quad , \quad k = K \cdot I \ \text{(assume)}$$

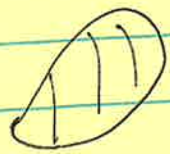
suppose P^1 element, ∇N is constant $\Rightarrow K^e_{ab} = (\nabla N_b \cdot \nabla N_a \cdot K_e) \underset{\sim}{A_e}$



$\longrightarrow V = A_e \cdot h \cdot \frac{1}{3}$ $\quad \rightsquigarrow f_e \int_{\Omega_e} N^e_a \, d\Omega = \left(\frac{1}{3} f_e \cdot A_e\right) \cdot 1$

02/20/2025.

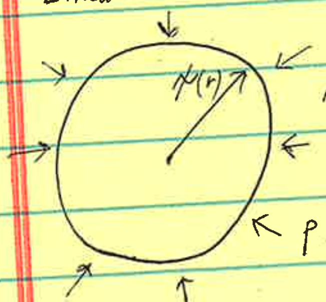
- Minimum Energy Principle. (at equilibrium).

 $\rightarrow$ Linear Elasticity.

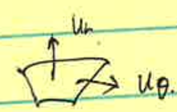
$$\left.\begin{aligned} \nabla \overset{\nearrow \text{force.}}{f} &= (\partial_1 f, \partial_2 f) \\ \underset{\downarrow \text{displacement.}}{\nabla u} &= \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix} \end{aligned}\right\} \text{ so that } \underset{\downarrow \text{strain}}{\varepsilon} = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

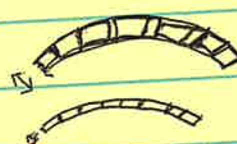
$\operatorname{div}(u) = tr(\nabla u)$, $A : B = A_{ij}B_{ij} = tr(AB^T)$

- Linear Elasticity is minimizing $V(u) \downarrow$ in equilibrium.



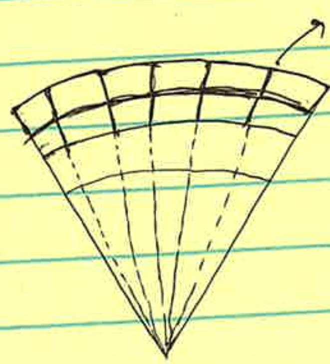
$$\left.\begin{aligned} \psi(r)\,\hat{e}_r &= u_r \\ 0 &= u_\theta \end{aligned}\right\} \rightarrow \text{why ?}$$





$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$, $\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$.

$\varepsilon_{r\theta} = \sim\sim$.



Along this line, $u_\theta = 0$

$\Rightarrow u_\theta = 0,\ u_r \lessgtr 0$.

- Weak form

$$\beta^{-1} \Delta q - \nabla U \cdot \nabla q = 0 \quad \rightarrow \quad \beta^{-1} q'' - U' q' = 0 \quad \text{for 1D.}$$

$$\Rightarrow \int \left(\beta^{-1} q'' - U' q' \right) v \, dx = \left[\beta^{-1} q' v \right] - \int \beta^{-1} q' v' \, dx - \int U' q' v \, dx = 0.$$

Let $v = 0$ at ∂A and ? $\rightarrow$ Use another method.

- Minimizing Elastic Energies.

$V(w) = \frac{1}{2} a(w, w) - \ell(w)$. $\rightarrow$ Exists Minimizer $\underline{V(u) \leq V(w)}$.

Ex) $\sigma = \dfrac{E}{1+\nu} \varepsilon(\nabla u) + \dfrac{E\nu}{(1+\nu)(1-2\nu)} (\nabla \cdot u) I$.

- Trick.

∇U

03/04/2025

- Cea's Lemma $u_h, v_h \in S_h$. $w_h \in V_h$

$$\left.\begin{aligned} a_h(u_h, w_h) &= l_h(w_h) \\ a_h(v_h, w_h) &= l_h(w_h) \end{aligned}\right\} \Rightarrow a_h(u_h - v_h, w_h) = 0$$

Note $u_h - v_h \in V_h$ (affine subspace).

$\Rightarrow a_h(u_h - v_h, u_h - v_h) = 0$ ——— (1)

By coercivity, there exists $\alpha > 0$ s.t

$a_h(v_h, v_h) \geq \alpha \, \|v_h\|^2$ (for $v_h \in V_h$)

From (1), $0 = a_h(u_h - v_h, u_h - v_h) \geq \alpha \, \|u_h - v_h\|^2 \geq 0$

$\Rightarrow u_h = v_h \rightarrow$ uniqueness.

$\therefore$ Coercivity implies uniqueness.

· Continuity

$|a_h(u - w_h, v_h)|$
$\leq M \, \|u - w_h\| \, \|v_h\|$

$|l_h(v_h)| \leq m \, \|v_h\|$

* How uniqueness imply existence.

$KV = F$. recall rank-nullity theorem,

· Fin. Elem sol exists, unique, following a priori approximation holds

$$\|u_h - u\| \leq \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in S_h} \|u - w_h\|$$

Pf) $a_h(u_h - w_h \overset{\substack{V_h \\ \Psi}}{,} u_h - w_h) \geq \alpha \|(u_h - w_h)\|^2$

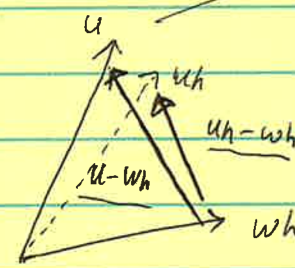
Also, $a_h(u_h - w_h, u_h - w_h) \leq a_h(u_h - u, u_h - w_h)$ Consistency!

$\qquad + \; a_h(u - w_h, u_h - w_h)$

Recall $a_h(u_h - u, v_h) = 0$ ∵ Galerkin orthogonality

$$\left(\begin{array}{l} \because \; a(u, v_h) = \ell(v_h) \\ - \; a(u_h, v_h) = \ell(v_h) \\ \hline a(u - u_h, v_h) = 0 \end{array} \right.$$

$$\Rightarrow \alpha \|u_h - w_h\|^{\cancel{2}} \leq a_h(u - w_h, u_h - w_h) \leq M \|u - w_h\| \cdot \cancel{\|u_h - w_h\|}.$$



$$\Rightarrow \|u_h - w_h\| \leq \frac{M}{\alpha} \cdot \|u - w_h\|$$

$\|u - u_h\| \leq \|u - w_h\| + \|u_h - w_h\|$ (triangle)

$$\Rightarrow \|u - u_h\| \leq \left(1 + \frac{M}{\alpha}\right) \cdot \min \|u - w_h\|$$

03/04/2025.

- What is norm?

① $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$

② $\|\alpha v\| = \alpha \|v\|$

③ $\|u+v\| \leq \|u\| + \|v\|$

* H_1 - norm.

$$\|v\|_{1,2} = \left(\int_a^b v^2 \, dx + \int_a^b v'^2 \, dx \right)^{1/2}$$

$$= \left(\underbrace{\|v_{0,2}\|^2}_{L_2 \text{ norm}} + \underbrace{\|v\|_{1,2}^2}_{H_1 \text{ - seminorm}} \right)^{1/2}$$

L_2 norm space $= \{ v : \Omega \to \mathbb{R} \mid \|v\| < \infty \}$

↳ should be square integrable.

H_1 norm space $= \{ v : \Omega \to \mathbb{R} \mid \|v\| < \infty \}$

EX) $f(x_1, x_2) = \log(1+x_1) + \log(1+x_2)$

For L_2 - norm, check square integrable. $\Rightarrow f \in L^2(\Omega)$

For H_1 - norm, $\partial f / \partial x_1 = \frac{1}{1+x_1} \uparrow \infty$ at $x_1 = -1 \Rightarrow f \notin H^1(\Omega)$

- Continuity

  $|a(u,v)| \leq \|u'\|_{L^2(0,1)} \|v'\|_{L^2(0,1)}$

- Coercivity

  $\int_0^1 |u'(x)|^2 dx \geq \alpha \|u\|^2_{H^1(0,1)}$

Cea's lemma

$\Rightarrow \|u - u_h\|_{H^1} \leq \left(1 + \frac{M}{\alpha}\right) \cdot \min_{v_h \in V_h} \|u - v_h\|_{H^1}$

- Convergence Rate.

  $\rightarrow$ Poisson type, P^k element, h mesh size

H_1 - seminorm $\|u - u_h\|_{H_1} = O(h^k)$

L^2 - norm " $= O(h^{k+1})$

L^1 - norm $= O(h^{k+1})$

Doesn't depend on the dimension