Addition + Multiplication → . Needed to form $\boxed{\text{linear combination}}$.

V.S → $\mathbb{C}$ (복소수)

① Addition.

$z_1 + z_2 = z_2 + z_1$ (commutativity)

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (associativity)

$z + 0 = z$ (additive identity)

$z - z = 0$. (additive inverse).

4 properties.

② Multiplication (scalar multiplication).

$\alpha(z_1 + z_2) = \alpha z_1 + \alpha z_2$ (distributive) — I.

$(\alpha_1 + \alpha_2) z = \alpha_1 z + \alpha_2 z$ (distributive) — II

2○.

What is Vector Space?

(informal def.) V.S = any set of objects. s/t linear comb. of set is in the set.

(formal def.) A set is a real Vector space. (V.S.) if there is.

addition operation '+' and scalar mult. '$\times$' s/t for any $v, w \in V$

1) $v + w = w + v$ (commutativity)

2) $(v+w)+u = v + (w+u)$ (associativity)

3) there is an element '0' s.t. $v + 0 = v$ (additive identity)

4) $v - v =$ '0' (additive inverse)

5) for any $\alpha, \beta \in \mathbb{R}$ (can't be $\alpha, \beta \in \mathbb{C}$?)

$(\alpha + \beta) v = \alpha v + \beta v$.

$\alpha(v + w) = \alpha v + \alpha w$. (distributive prop.)

Ex) ① $\mathbb{R}, \mathbb{C}, \mathbb{R}^n$

② $\{0\}$

③ {All m×n matrix} including zeros (m×n)

④ Set of polynomials of degree up to n

Q: How many elements in a V.S?

A: 1 to ∞

$\{0\}$

What is vect.

1) Col(A) = set of all linear combinations of columns.

2) row(A) = " " rows.

3) null(A) = $V = \{x \mid Ax = 0\}$.

= $N(A)$

How to check?

① $\{0\}$

② $\alpha x + \beta y$

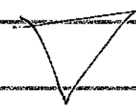
10/20/23.

ME 300A — Rank, Vector Space, ..

• Rank-Nullity Theorem.

rank(A) + dim(null(A)) = n

-1 $\quad 0 \quad 1$ $\qquad 1 \; 2$ $\qquad$ FCC $\quad$ BCC



E.g. 1 $A = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}$ $\quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

① Find basis for null(A).

col(A) $\rightarrow \begin{pmatrix} +2 \\ +1 \\ 0 \end{pmatrix} \begin{pmatrix} +1 \\ +1 \\ +1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

row(A). $(1 \quad 0 \quad -1)$, $(0 \quad 1 \quad 2)$

② rank(A) = 2

Fill.

E.g. 2. $Ax = b$ when $b = \begin{pmatrix} 0 \\ -1/2 \\ -1 \end{pmatrix}$ $\qquad a \quad -2a \quad \frac{2a+1}{2}$

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1/2 \\ 0 & -1 & -2 & -1 \end{array}\right] \rightarrow \begin{array}{cccc} -2 & -2 & -2 & -1 \\ -1 & -1 & -1 & -1/2 \\ 0 & -1 & -2 & -1 \end{array}$$

$\frac{1}{2}$ $\quad 3$

$-4/2 \quad -2 \rightarrow$ no solution

• pylammps. $\Leftrightarrow$ lammps with python — binary on own computer

10/27/23

Determinant.

Def) scalar function of matrix entries that can determine whether the matrix is singular. (Note: computational work heavy).

~~Note~~: How to determine?

① $1\times1 \rightarrow \det([A]) = A$

② $\det\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{1+1} a_{11} \det(a_{22}) + (-1)^{1+2} a_{12} \det(a_{21})$

$\cancel{(-1)^{2+1} a_{21}} \quad = a_{11}a_{22} - a_{12}a_{21}$

③ $\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} = \boxed{\sum_{i,j} (-1)^{i+j} a_{ij} \, |C_{ij}| \quad (\text{for } 1 \le i \le n)}$

C_{ij} is A with i^{th} row & j^{th} col eliminated.

• E.g) $\det\begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0.$ (∵ column of zeros)

number of row flip.

det(A) = 0 iff A is singular → $\det(A) = (-1)^{k} \det(G.E.(A))$

$\{E_{b\,i}, E_{b\,i+1} \cdots E_{b\,j-1}, E_{b\,j}\}$ → Batch

(length: $j-i+1$)

Update, $K_{i\to i+1}$

$K_{i+1\to i}$ $\quad K_{i+1\to i+2}$

$K_{i+2\to i+1}$ $\quad K_{i+3\to i+3}$

⋮

$K_{j-1\to j}$ $\quad K_{j-1\to j-2}$

$K_{j\to j-1}$

$b \in col(A) \Leftrightarrow Ax=b$ has a solution

$\dim(N(A)) = n - rk(A)$

tells you # of D.O.F in solutions of system.

$\Rightarrow Ax = \vec{0} \Leftrightarrow Ux = 0 \Rightarrow \sum x_i \vec{u_i}$ + a same relation btw col(A) & col(U)

$\Rightarrow \dim(col(A)) + \dim(N(A)) = n$

☆ Every vector $\in \mathbb{R}^n$ is sum of two components (vectors) in col(A) and N(A)

10/30/23

⟨ Orthonormal bases + QR factorization ⟩

Note : $A\vec{x} = \vec{b}$ ① { solution exists iff $b \in col(A)$
$x_1 \vec{a_1} + \cdots + x_n \vec{a_n} = b$. }

② $Null(A)$ determines # of solutions

D.O.F (Free params) $= \dim(N(A)) = n - rank(A)$

Q) How to find coeffs? $(a_1 \cdot x_1 + \cdots + a_n x_n = \vec{b_n}) \Rightarrow (a_1 \sim a_n)$

A) Hard but if we have ortho-normal basis of col(A) → easy

- Orthonormal = ①orthogonal + ②normal.

{ ① : $V^T \cdot w = 0$ → Perpendicular (dot product = 0)

② $\|V\| = 1$, $\|w\| = 1$

Def) The vectors $\vec{q_1} \sim \vec{q_n}$ is orthonormal if they are mutually orthogonal and each vectors have norm = 1

$\Rightarrow q_i^T q_j = \delta_{ij}$, $\|q_i\| = 1$

- Suppose A is non-singular and wat to solve $Ax = b$
- If $q_1 \sim q_n$ is orthonormal basis of col(A), then

$c_1 \vec{q_1} + \cdots + c_n \vec{q_n} = \vec{b}$

$\Rightarrow \vec{q_i}^T (c_1 \vec{q_1} + \cdots c_n \vec{q_n}) = \vec{q_i}^T \cdot \vec{b}$

$\Rightarrow c_i q_i^T q_i = c_i = q_i^T \cdot \vec{b}$ → easy to find coeff.!

$$\begin{bmatrix} \vec{q_1} & \cdots & \vec{q_n} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}$$

$\rightarrow Q \cdot \vec{c} = \vec{b}$

$$\therefore \vec{c} = \begin{pmatrix} q_1^T b \\ \vdots \\ q_n^T b \end{pmatrix} = Q^T \cdot \vec{b}$$

Note that $Q \cdot \vec{c} = \vec{b}$ $\quad Q^T \cdot \vec{b} = \vec{c}$ $\Leftrightarrow$ $Q^T \cdot Q = I$
$\qquad \| $
$\qquad Q \cdot Q^T$

- $Q^{-1} = Q^T$ (orthonormal). Ex: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ → Hadamard gate.

A $n \times n$ matrix is $\boxed{\text{orthogonal}}$ if columns are $\boxed{\text{orthonormal}}$
⇒ $n \times n$ Q is orthogonal iff $Q^T Q = I$.

- How to make Q? → Gram-Schmidt Process.
  Caveat: not good for numerics (in practice)

<G. S. process>

→ In each step, use new column of A and remove components parallel to (contained in span of) vectors you've already found.

① $\vec{q_1} = \vec{a_1} / \|\vec{a_1}\|_2$ → Euclidean norm.

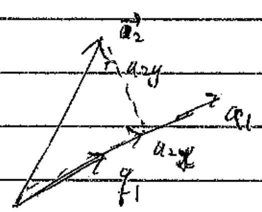
② we want $\vec{a_2} \in \text{span}(\vec{q_1}, \vec{q_2})$ ⇒ $\vec{a_2} = c_1 \vec{q_1} + c_2 \vec{q_2}$

⇒ $\vec{q_1} \cdot \vec{a_2} = c_1 \cdot \vec{q_1} \cdot \vec{q_1} + \cancel{c_2 \vec{q_2} \cdot \vec{q_1}} \to 0$

⇒ $\vec{q_1}^T \cdot \vec{a_2} = c_1$ — (1)

using (1), $c_2 \vec{q_2} = \vec{a_2} - (\vec{q_1}^T \cdot a_2) \cdot \vec{q_1}$.

$$\vec{w_2} = \vec{a_2} - (\vec{q_1}^T \cdot \vec{a_2}) \vec{q_1} \Rightarrow \vec{w_2} / \|\vec{w_2}\|_2 = \vec{q_2}$$



$q_1 (\vec{q_1}^T \cdot \vec{a_2}) = \vec{a_{2x}} = \text{proj}_{\vec{q_1}}(\vec{a_2})$

(∵ $\vec{q_1}$ is already normalized)

⇒ $\vec{q_2}$ = normalized $\vec{a_{2y}}$ ⊥ $\vec{a_1}$

③ $a_3 \in \{q_1, q_2, q_3\}$ ⇒ $\vec{a_3} = c_1 \vec{q_1} + c_2 \vec{q_2} + c_3 \vec{q_3}$ Continue..

⇒ $c_3 \cdot \vec{q_3} = \vec{a_3} - c_1 \vec{q_1} - c_2 \vec{q_2} = \vec{a_3} - \vec{q_1}^T \vec{a_3} \cdot \vec{q_1} - \vec{q_2}^T \vec{a_3} \cdot \vec{q_2}$

⟨G.S. process⟩

$\Rightarrow$ At $\vec{a_k} = \text{span}\{\vec{q_1} \cdots \vec{q_k}\}$ (we want this)

$\Rightarrow \vec{a_k} = c_1 \vec{q_1} + \cdots + c_k \vec{q_k}$ ..

Note that $c_1 = \vec{q_1}^T \cdot \vec{a_k}$ $\cdots$ $c_{k-1} = \vec{q}_{k-1}^T \cdot \vec{a_k}$

$\Rightarrow c_k \vec{q_k} = \vec{a_k} - (\vec{q_1}^T \cdot \vec{a_k}) \cdot \vec{q_1} - \cdots - (\vec{q}_{k-1}^T \vec{a_k}) \vec{q}_{k-1}$

$$\boxed{c_k \vec{q_k} = \vec{a_k} - \sum_{j=1}^{k-1} (\vec{q}_{k-1}^T \cdot \vec{a_k}) \cdot \vec{q}_{k-1}}$$

$\vec{w_k} \rightarrow$ Normalize $\rightarrow$ $\vec{q_k} = \vec{w_k} / \|\vec{w_k}\|_2$

☆ Note : $\vec{q_1}^T \cdot \vec{a_k}$, $\vec{q_2}^T \cdot \vec{a_k}$, ... , $\vec{q}_{k-1}^T \vec{a_k}$

We obtain $\Rightarrow A = Q \cdot R$ where Q orthogonal & R upper triangular

$\Rightarrow r_{ik} := \vec{q_i}^T \cdot \vec{a_k}$

Ex.) $\begin{bmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{bmatrix} = Q \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$

$\Rightarrow \vec{a_2} = r_{12} \vec{q_1} + r_{22} \vec{q_2}$

11/08/23.

- Eigenvectors & Eigenvalues.

- Square matrix = operators.

$$\begin{array}{ccc} \mathbb{R}^n & & \mathbb{R}^n \\ (V) & \xrightarrow{A} & (U) \\ \{\vec{v}\} & & \{\vec{u}\} \end{array}$$

There are some vectors $A\vec{x} = \lambda\vec{x}$

1. Eigenvalue (λ)
2. Eigenvector ($\vec{x}$)

⇒ Def) $Ax = \lambda x$ satisfying $x \neq 0 \rightarrow v, \lambda =$ (eigenvector eigenvalue)

- solve $(\lambda I - A)\vec{x} = 0 \iff (\lambda I - A)$ must be dependent

only possible if $(\lambda I - A)$ is singular $\equiv$ eigenvector/value exists.

$\equiv \boxed{\det(\lambda I - A) = 0} \rightarrow$

E.g.) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \left| \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} \right| = 0 \Rightarrow \lambda = \pm 1$

$\Rightarrow$ ① $\lambda = 1 \quad Ax = x \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \vec{v}_1$

② $\lambda = -1 \quad Ax = -x \Rightarrow \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \vec{v}_2$

spans!

$\vec{v}_1 \in \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $\vec{v}_2 \in \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ → non-unique

$\vec{v}_1, \vec{v}_2$: eigenvector for λ $\Rightarrow c_1 A\vec{v}_1 + c_2 A\vec{v}_2$

$= c_1 \lambda \vec{v}_1 + c_2 \lambda \vec{v}_2$

$= \lambda(\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2}_{\boxed{\text{new eigenvector}}})$

⟨new eigenvector⟩

Therefore, eigenvector for one ~~eigenvalue~~ eigenvalue is not unique!

Q) Given eigenvalue λ, of A, what dim of subspace of corresponding eigenvectors?

A: num. of linear independent λ - eigenvalues is $\dim(N(\lambda I - A))$

E.g. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$I\vec{v} = \vec{v}$ and $J\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow 1$ is eigvalue of I, J

① ⇒ How many lin. independent? ⇒ $(I - 1 I)v = 0 \Rightarrow$ 2

② $(J - I)\vec{v} = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\vec{v} = 0 \Rightarrow$

$\dim(N(J-I)) = 2 - rk(J-I) = 2 - 1 = 1$

⇒ 1 lin. independent vectors

Key point

there are always 'n' eigvalues

11/10/23

Q a) A is singular $\Leftrightarrow$ 0 is eigenvalue of A

True) pf) ($\Rightarrow$) A is singular $\rightarrow$ $v \neq 0$ s/t $Av = 0$ $\rightarrow$ eigenvalue

($\Leftarrow$) $Av = 0$ $\rightarrow$ A is singular

Alternate $\rightarrow$ $0 = \det(A) = \det(0I - A) = 0 \Rightarrow 0$ is eigenvalue

b) A and A^T has same eigenvalues.

True $|(\lambda I - A)| = |(\lambda I - A^T)|$ no diagonal changes.

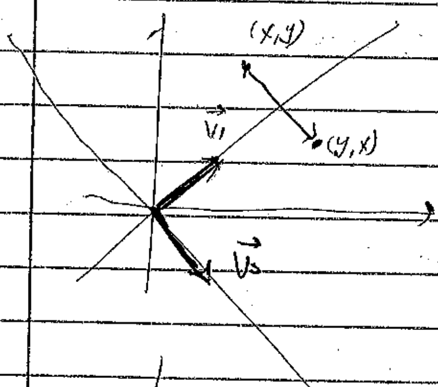
$$\det(\lambda I - A) = |\lambda I^T - A^T| = |\cancel{(\lambda I - A)^T}|$$

$$= |\lambda I - A^T| \quad // \quad \#$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \lambda = -1, 1 \quad \vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

what does this transformation look? $\rightarrow$ switch basis (reflection)

$(x,y) \rightarrow (y,x)$



Upon reflection, eigenvectors

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow$ stays same

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow$ reflected.

This is householder reflection from HW3.

$$Av = I - \frac{2}{v^T v} v v^T$$

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

① Recall that $\lambda(z) = \det(A - zI)$ has degree (n)
for $n \times n$ matrix A with n eigenvalues.

② Eigenvectors corresponding to distinct eigenvalues are linearly independent.
why ?

③ $[A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n] = [\lambda_1 \vec{v}_1 \ \ \lambda_2 \vec{v}_2 \ \cdots \ \lambda_n \vec{v}_n]$

$$A \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix}$$

$$\Rightarrow A \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{V} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{\Lambda}$$

$\Rightarrow AV = V\Lambda$ is V invertible $\Rightarrow$ Yes, (v_i, v_j independent)

< canonical form - factorization >

$\Rightarrow$

* Degenerate cases (repeated eigenvalues)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

Repeated eigenvalues but not diagonalizable ?

$\Rightarrow J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Eigvals = (1), only one eigvec $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ Not enough eig-vecs to form basis.

- Multiplicity of eigenvalue λ = mult of root of charact. polyn.

$\Rightarrow$ Geometric multiplicity = $\boxed{\text{\# of lin. ind. } \lambda\text{s}}$.

- ODE

$\vec{y}' = A\vec{y}$

$$\Rightarrow \vec{y} = e^{At} \cdot \vec{y}(0)$$

$\rightarrow$ Canonical form makes this computation simple!

11/13/23

- Q & A session.

- Least squares arises when $A\vec{x} = \vec{b}$ has
<u>no solution</u> or too many solution.

→ when system is inconsistent $\left\{ \begin{array}{l} \text{mostly,} \\ \underline{\text{overdetermined}} \end{array} \right\}$

→ when no way to meet all requirements...

- $A\vec{x} = \vec{b}$ ⇒ $\vec{b}$ doesn't belong to col(A) ⟺ $\vec{b} \notin col(A)$

⇒ $(A\vec{x} - \vec{b})^T (A\vec{x} - \vec{b}) = l(\vec{x})$

$\nabla l(\vec{x}) = 0$ ⇒ $A^TA\vec{x} = A^T\vec{b}$ → (As long as A's column independent, it has solution)

⇒ If A has lin. ind. col. → $A^TAx = b$ has solution
(Even if $Ax = b$ doesn't have solution)

Why?) It means A full rank = $N(A) = 0$

Note that $N(A) = N(A^TA)$ for any A (det(A^TA), and det(A) non zero.)

pf) $N(A) = \{x \mid A\vec{x} = 0\}$ → $A^TAx = A^T0 = 0$

$N(A^TA) = \{x \mid A^TAx = 0\}$ →

⇒ A^TA is invertible ⇒ $X_{ls} = (A^TA)^{-1} A^Tb$

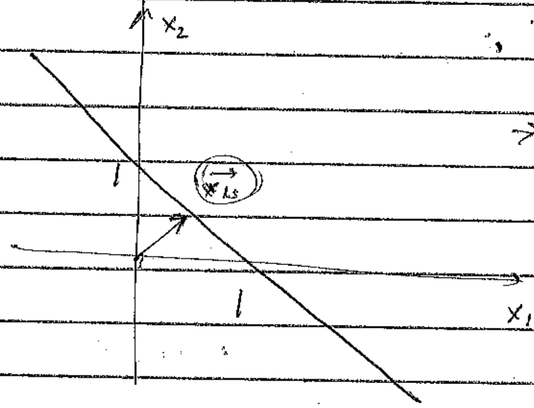
The cond. # $k(A^TA) = (k(A))^2$ of coeff. mat for normal eigenvalues can be much bigger than that of A.

Fix: Use QR factorization.

Least norm solution.

Typically underdetermined when we want to ~~[illegible]~~ select a certain infinitely many $\rightarrow \underset{\vec{x}}{\text{minimize}} \|\vec{x}\|_2$

eg) $A = [1 \quad 1] \quad b = [1] \rightarrow x_1 + x_2 = 1$



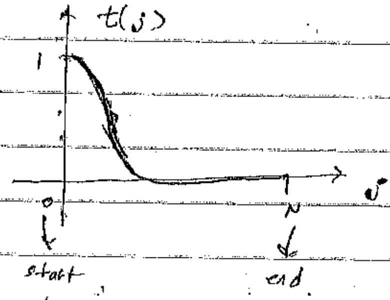
$\rightarrow$ choose closest to origin.

$$\underline{\vec{X}_{LN} = A^T (AA^T)^{-1} \vec{b}}$$

11/21/23.

Lecture 13.

① $K'(i \to j) = K(i \to j) \cdot \{1 - t(j)\}$



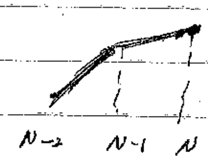
$K'(i \to 0) = 0.$

②

$n'(i) = 1$ (for $i \neq N$)

$P_s(i) = \sum_{j \neq 0} K(i \to j) P_s(j)$

$\Rightarrow P_s(N) = \sum_{j \neq 0} \underline{K(N \to j)} \cdot P_s(j)$,



$= K(N \to N-1) \cdot P_s(N-1)$ (for 1D), if N is final (no $N+1$)

what happens when we consider $n'(N) = 1$

11/27/23.

Lecture 13

Computing eigenvalues.

- consider diagonalizable matrix. $A \in \mathbb{R}^n \times \mathbb{R}^n$. $|\lambda_1| > \cdots |\lambda_n|$

given $u^{(0)} \rightarrow u^{(1)} = A u^{(0)} \rightarrow u^{(2)} = A u^{(1)} = A^2 u^{(0)}$

$$\Rightarrow u^{(k)} = A^k \cdot u^{(0)}$$

Note. $u^{(0)} = \sum_{i=1}^{n} \alpha_i \vec{v_i}$ — Basis of $\mathbb{R}^{n\times n}$

$$\Rightarrow A u^{(0)} = \sum_{i=1}^{n} \alpha_i \lambda_i \vec{v_i}$$

$$\Rightarrow u^{(k)} = \sum_{i=1}^{n} \alpha_i \lambda_i^k \vec{v_i} = \lambda_1^k \cdot \sum_{i=1}^{n} \alpha_i (\lambda_i / \lambda_1)^k \vec{v_i}$$

As $k \rightarrow \infty$, $\underline{u^{(k)} \approx \lambda_1^k \cdot \alpha_1 \vec{v_1}}$

→ consider symmetric, diagonalizable matrix. $A \in \mathbb{R}^{n\times n}$.
$(\lambda_1 > \cdots \lambda_n > 0)$

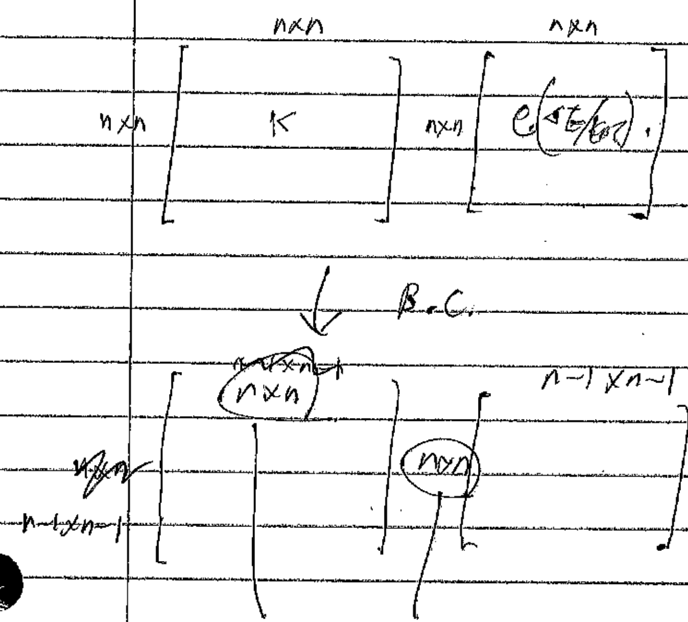
A canonical decomposition. $A = V \Lambda V^{-1} = V \Lambda V^T$

$$\rightarrow A = \sum_{i=1}^{n} \lambda_i \vec{v_i} \vec{v_i}^T$$

11/29/23.

QR iteration.

Characteristic polynomial : $\det(zI - A) \longrightarrow$ (II)



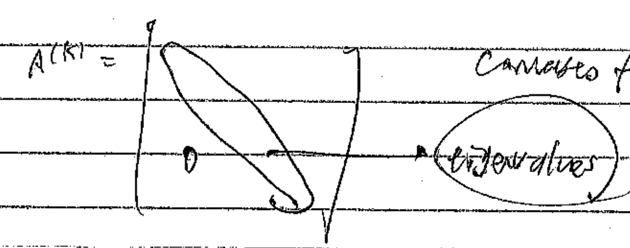
$$A^{(1)} = Q^T A Q = Q^{-1} A Q$$

$$\downarrow$$

$$A^{(k)} = \{Q^{(k-1)} \cdots Q^{(1)}\} A \{Q^{(1)} \cdots Q^{(k-1)}\}$$

< Iterative QR >

A is real, non-singular → Converge



12/4/23.

C-H (Cayley-Hamilton) Theorem, SVD.

- QR iteration.

$\Rightarrow$ while converge

$$\begin{cases} Q, R = qr(A) \\ A \leftarrow RQ \end{cases} \rightarrow$$

Upper triangular matrix

Eigenvalues are on diagonal entries.

- C-H Thm.

$\Rightarrow$ Every matrix satisfies its char. poly. $\chi_A(A) = 0_{n \times n}$

$\chi_A(z) = \det(A - zI)$ ~~and~~ denoting

$$\Rightarrow \chi_A(z) = (z - \lambda_1) \cdots (z - \lambda_n)$$

$$= z^n + c_1 z^{n-1} + \cdots + \det(A)$$

$\lambda_1 \cdots \lambda_n$.

pf. To make it easy, suppose A is diagonalizable, $A = T \Lambda T^{-1}$

$$\left.\begin{aligned} A - \lambda_k I &= T \Lambda T^{-1} - \lambda_k I = T \Lambda T^{-1} - T \lambda_k I T^{-1} \\ &= T(\Lambda - \lambda_k I) T^{-1} \end{aligned}\right\} \text{Supp.}$$

Note that,

$\chi_A(A)\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$

and also

A is diagonalizable so eigenvectors $\underline{\vec{v_1} \sim \vec{v_n}}$ are basis of $\mathbb{R}^n$.

$$\Rightarrow \vec{x} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$

we know $A\vec{v_j} = \lambda_j \vec{v_j} \iff (\lambda_j \cdot I - A)\vec{v_j} = 0$

now, we know that $(\lambda_i I - A).\vec{v_i} = 0$ where

$$\chi_A(A) = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I)$$

Continued, $\chi_A(A) = (A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$

using $(\lambda_j I - A)\vec{v_j} = 0$, for all v_j there exists
$(A-\lambda_j I)\vec{v_j} = 0$, exists. $\Rightarrow$ $\underline{\chi_A(A)\vec{x} = 0}$.

$\therefore$ $\underline{\chi_A(A) = O_{n\times n}}$ #

Now let's see general matrix A.

Q) If matrix is non-diagonalizable → still C-H works ($\because$ char. poly has enough $\boxed{\text{repeated factors}}$)

E.g.) . $J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ eig vals : 2, 2 $\therefore \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{rank } 1} x = 0$ eigenvalues

eig. vecs : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\chi_J(z) = (z-2)(z-2)$

→ $(J-2I)(J-2I)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

→ It has enough polynomials to make non-eigenvector to eigenvector so that $J\vec{c} = 0$

changed to eigvector from eig non eig-vector.

#

June

- Applications of Cayley – ~~Hamilton~~ theorem

$\chi_J(J) = (J-2I)^2$

$= J^2 - 4J + 4I = 0 \quad \Rightarrow \quad J(J-4I) = -4I$

$\Rightarrow \quad J - 4I = -4J^{-1}$

$\Rightarrow \quad J^{-1} = I - \frac{1}{4}J$ —— ①

Note: $\det(A - zI)\Big|_{z=0} = \chi_A(0) = \det(A)$ should be non-zero for ① to work

$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ but by C-H, $\chi_A(z) = z^n + \cdots + c_n$,

$\chi_A(A) = A^n + c_1 A^{n-1} + \cdots + c_n I = 0$

use this dependence to express series using finite-many terms.

$e^A = I + A + \cdots + \frac{A^n}{n!}$

$= I + (1 - c_{n-1})A + \left(\frac{1}{2} - c_{n-2}\right)A^2 + \cdots + \left(\frac{1}{(n-1)!} - c_1\right)A^{n-1}$

$+ \cdots$

$= k_1 I + k_2 A + \cdots + k_n A^{n-1}$ w/ $k_1 \sim k_n$ by series.

→ $\boxed{\text{Useful for Krylov subspaces}}$

SVD.

<Spectral decom>

$A \in \mathbb{R}_{n \times n}$ & $v_1 \sim v_n$ form basis of $\mathbb{R}^n$

$\Rightarrow A = T \Lambda T^{-1}$ $\qquad A\vec{v_j} = \lambda_j \vec{v_j}$

<Singular decom>

A is $m \times n$ $\Rightarrow$ Thm.) There is an orthogonal matrix $U(m \times m)$

an orthogonal matrix $V(n \times n)$ and

diagonal Σ $(m \times n)$ w/ non-negative entries.

$$A = U \Sigma V^T$$

12/8/23

Final Review

- Matrix operations
- Gauss elim + LU
- Gram-Schmidt + QR
- Vector space + basis
- Eig vec / Eig val.
- Decoupling mat. exponential.
- C-H theorem
- SVD

Not on final

- Iterative method.
- non-linear system
- gradient steep descent
- power method
- QR iteration

Recall)

An $\boxed{n \times n}$ matrix is $\boxed{\text{singular}}$ iff (if and only if)

① $\det(A) = 0$

② non-invertible $\equiv$ A^{-1} does not exist

③ A's columns are dependent.

④ $Ax = 0$, $N(A)$ contains other than 0 (non zero solution).

⑤ $Ax = 0$ for some non-zero x

⑥ $N(A) \neq \{0\} \Leftrightarrow \dim(N(A)) \geq 1$

⑦ $\text{rank}(A) < n$

⑧ 0 is an eigenvalue of $A \Leftrightarrow$ there is an eigenvector w/ eigval zero.

$\Leftrightarrow Av = 0$

⑨ ~~Number~~ At least (one) singular value is zero

$Au_i = \sigma_i v_i \rightarrow$ at least $\sigma_n = 0$

Recall) If A is $n \times n$ then λ is an eigenvalue of A iff

① $A\vec{v} = \lambda\vec{v}$ ($\vec{v} \neq 0$)

② $\det(\lambda I - A) = 0$

③ λ is a root of $\underbrace{\chi_A(\lambda) = \det(\lambda I - A)}_{\text{characteristic polynomial}} = 0$

④ Matrix $\lambda I - A$ is singular.

↳ ① $\Rightarrow A\vec{v} - \lambda\vec{v} = 0 \Rightarrow (\lambda I - A)\vec{v} = 0$

Recall) A is $n \times n$, then $e^A = I + A + \frac{1}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$

$\Rightarrow$ $A = U \Lambda U^T$ (Λ is diagonal matrix) : Diagonalize!

Assuming A is diagonalizable : (n) distinct eigenvalues, symmetric.

→ Note : this is from schur decomposition.

Ex.) Non-diagonalizable. Good Counter Example.

$J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: Full rank, Upper triangular : $\lambda_1 = 1$, $\lambda_2 = 1$

→ Non-diagonalizable

If diagonalizable, ①$\Leftrightarrow$ (If and only if) n independent eig-vectors $\vec{v_1} \sim \vec{v_n}$

$\Rightarrow$ ② $v_1 \sim v_n$ are basis of $\mathbb{R}^n$ $\Rightarrow$ ③ $T = [v_1 \cdots v_n]$

$\Rightarrow$ $A = T \Lambda T^{-1}$

$$\text{Note } A^2 = T\Lambda \cancel{T^{-1}} \cdot \cancel{T}\Lambda T^{-1} = \underset{\sim\sim\sim}{T\Lambda^2 T^{-1}}$$

e+2-1

$$\Rightarrow e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}(T\Lambda T^{-1})^k = \sum_{k=0}^{\infty} T\Lambda^k T^{-1}$$

$$= T\left(\sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k\right)T^{-1} = \underline{T e^{\Lambda} T^{-1}} = T\begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n}\end{pmatrix}T^{-1}$$

↳ can calculate.

Mu Ltrx
Matrix
✓ Cw-k
Matrix
Diagonal
Matrix.
Dru
Dia DD
DDD

- Symmetric matrix. — special, ($n \times n$ matrix)

E.

$\det(\lambda I - A) = \det(\lambda I - A^T)$

Recall — ⟨spectral theorem.⟩

If A is $n \times n$ symmetric, A has 'n' real eig vals. and 'n' orthonormal eig vecs.

$$\Rightarrow \underline{A = V\Lambda V^T}, \text{ where } V^T V = I = VV^T.$$

- Square matrix is orthogonal $\Leftrightarrow$ columns are orthonormal vectors.

Numerical methods.

optimizations
- CME 305
- EE 364

Lin. Alg.

Theory
- MATH 113
- MATH 104

Numerics
- CME 302
- EE 263

Applications
- Diff. Eq.
- CME 204
- ME 300B
- CME 306

Q1) For any $n \times n$ matrix A, e^A is non-singular.

$e^A = I + A + \frac{1}{2!}A^2 + \cdots$ → True

for $x \neq \vec{0}$, $Ax = 0 \Rightarrow$

if A is ~~non~~ diagonalizable. $A = T\Lambda T^{-1}$

$e^A = Te^{\Lambda}T^{-1} \Rightarrow (e^A)^{-1} = Te^{-\Lambda}T^{-1}$ → inverse exists!

How about general case?

$e^{-A} = I - A + \frac{1}{2!}(-A)^2 + \cdots$ $\left(I + A + \cdots \right)$ e^A

(e^A)

$e^A - Ae^A$

$I + \frac{1}{2!}$

$I + 1$

$I + A + \frac{1}{2!}A^2 + \cdots$

$-A - A^2 - \frac{1}{2!}A^3$

In general,

$(e^A)^{-1} = e^{-A} \Rightarrow e^{-A} \cdot e^A = e^{-A+A} = e^0 = I$

⇒ It holds b/c A and $-A$ commute.

~~In general,~~ $e^A e^B \neq e^{A+B}$

Show that if A and B commute, $e^A e^B = e^{A+B}$

12/11/23

Q) How to prove Hermitian matrix's eig. vals are real.

$(\hat{A} = \hat{A}^\dagger)$

suppose $\hat{M}|\alpha\rangle = \lambda_\alpha |\alpha\rangle \Rightarrow \langle\alpha|\hat{M}|\alpha\rangle = \langle\alpha|\lambda_\alpha|\alpha\rangle = \lambda_\alpha \cdot \langle\alpha|\alpha\rangle$

$= \lambda_\alpha$

where $(\hat{M}|\alpha\rangle)^\dagger = \langle\alpha|\hat{M}^\dagger = \langle\alpha|\lambda_\alpha^* \Rightarrow \langle\alpha|\hat{M}^\dagger|\alpha\rangle = \langle\alpha|\lambda_\alpha^*|\alpha\rangle = \lambda_\alpha^*$

since $\hat{M}^\dagger = M^\dagger$, $\langle\alpha|M^\dagger|\alpha\rangle = \langle\alpha|M|\alpha\rangle \Rightarrow \lambda_\alpha = \lambda_\alpha^*$ #

$\therefore \lambda_\alpha \in \mathbb{R}$.

SVD – singular Value Decomposition.

$$\underline{A = U\Sigma V^T}$$

$$\overset{V^T}{j\begin{bmatrix} \text{---} \\ \text{--- } v^T \text{ ---} \\ \text{---} \end{bmatrix}}\overset{\vec{v_j}}{\begin{bmatrix} \vec{v_j} \end{bmatrix}} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow j\text{th} = \vec{e_j}$$

$$A \cdot v_j = U\Sigma(V^T \cdot \vec{v_j}) = U\Sigma(\vec{e_j})$$

$$= \sigma_j U\vec{e_j} = \underline{\sigma_j \vec{u_j}}$$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_j & \\ & & & \ddots \\ & & & & \sigma_r \\ & & & & \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow j\text{th} = \begin{pmatrix} 0 \\ \vdots \\ \sigma_j \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \underline{A\vec{v_j} = \sigma_j \vec{u_j}} \quad \text{—} ①$$

$$A^T \vec{u_j} = (V\Sigma U^T)\vec{u_j} = V\Sigma \vec{e_j} = \sigma_j V\vec{e_j} = \sigma_j \vec{v_j}$$

$$\Rightarrow \underline{A^T \vec{u_j} = \sigma_j \vec{v_j}} \quad \text{—} ②$$

$$② \rightarrow ① \Rightarrow A^T \vec{u_j} = A^T \cdot \frac{1}{\sigma_j} A\vec{v_j} = \sigma_j \vec{v_j}$$

$$\left.\begin{array}{ll} \Rightarrow A^TA \cdot \vec{v_j} = (\sigma_j)^2 \vec{v_j} & (A^TA\text{'s eig.vec} = \vec{v_j}) \\ ① \rightarrow ② \Rightarrow AA^T \vec{u_j} = (\sigma_j)^2 \vec{u_j} & (AA^T\text{'s eig.vec.} = \vec{u_j}). \end{array}\right\} \text{—} (*)$$

$$\left.\begin{array}{l} A\vec{v_j} = \sigma_j \vec{u_j} \\ A^T\vec{u_j} = \sigma_j \vec{v_j} \end{array}\right\}$$

Let's say r non-zero singular values.

$$\Rightarrow A\vec{v_{r+1}} = A\vec{v_{r+2}} = \cdots = A\vec{v_n} = \underset{\overset{\|}{0}}{\sigma_{r+1}} \vec{u_j} = \underset{\|}{\sigma_r} = 0.$$

$$\Rightarrow \vec{v_{r+1}}, \vec{v_{r+2}}, \cdots, \vec{v_n} \in \text{null}(A) = N(A). \quad \text{—} ③$$

if $\sigma_j \neq 0$, $A\underset{\text{entire}}{\underset{\downarrow}{\vec{v_j}}} = \sigma_j \vec{u_j} \rightarrow$ expressible.

$\therefore \vec{u_1}, \cdots, \vec{u_r} \in \text{col}(A)$.

left singular eigenvectors

$$\begin{cases} A\vec{v_j} = \sigma_j \vec{u_j} \\ A^T\vec{u_j} = \sigma_j \vec{v_j} \end{cases} \qquad A = U\Sigma V^T$$

↳ left singular

~~right~~ singular eigen vectors.

Suppose $\sigma_1 \sim \sigma_r \neq 0$, $\sigma_{r+1} \sim \sigma_n = 0$.

① $\vec{u_1}, \cdots, \vec{u_r} \in col(A) \Rightarrow rk(A) \geq r$

$\vec{v_{r+1}}, \cdots, \vec{v_n} \in N(A) \Rightarrow \dim(N(A)) \geq n-r$

② $\vec{v_1}, \cdots, \vec{v_r} \in col(A^T) \Rightarrow rk(A^T) \geq r$

$\vec{u_{r+1}}, \cdots, \vec{u_n} \in N(A^T) \Rightarrow \dim(N(A^T)) \geq n-r$

①, ② $\Rightarrow$ $rk(A) = rk(A^T) = r$, $\dim(N(A)) = \dim(N(A^T)) = n-r$

$A^TA\vec{v} = \lambda\vec{v}$ I

$\|A\vec{v}\|^2 = \vec{v}^T A^TA\vec{v} = \vec{v}^T\lambda\vec{v} = \lambda\|\vec{v}\|^2$

$\Rightarrow \lambda = (\|A\vec{v}\| / \|\vec{v}\|)^2 \geq 0$

$A^TA =$

12/11/2023.

Review

- 1-norm $= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| = \|A\|_1$. $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

2-norm $= \max_{\vec{x}\neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \|A\|_2$

Frobenius norm $= \|A\|_F =$

1) $\boxed{Q \cdot R}$ $A = [\, a_1 | \cdots | a_n \,] \rightarrow [e_1 \sim e_n] = Q$

$$R = \begin{bmatrix} \vec{e_1}\cdot\vec{a_1} & \vec{e_1}\cdot\vec{a_2} & \vec{e_1}\cdot\vec{a_3} & \\ & \vec{e_2}\cdot\vec{a_2} & \cdots & \\ & & \ddots & \\ & & & \vec{e_n}\cdot\vec{a_n} \end{bmatrix}$$

2) If A is symmetric w/ distinct eig. vals. $(\lambda_i \neq \lambda_j)\ \forall i\neq j$

$\vec{v_i}^T A \vec{v_j} = \vec{v_i}^T(\lambda_j \vec{v_j}) \Rightarrow \vec{v_i}^T A \vec{v_j} = \vec{v_i}^T \lambda_j \vec{v_j}$

$\Rightarrow \vec{v_j}^T A^T \vec{v_i} = \vec{v_j}^T \cdot \lambda_j (\vec{v_i})$

$\Rightarrow \vec{v_j}^T A \vec{v_i} = \lambda_i \vec{v_j}^T \cdot \vec{v_i} = \lambda_j \vec{v_j}^T (\vec{v_i})$

$\Rightarrow \underline{(\lambda_i - \lambda_j)\ \vec{v_j}^T \vec{v_i}} \Rightarrow \because \lambda_j \neq \lambda_i \Rightarrow \vec{v_j}^T \cdot \vec{v_i} = 0 \ (i\neq j)$

3) Strictly diagonally dominant (SDD)

$\underline{|a_{ii}| > \sum_{j\neq i} |a_{ij}|}$ (for $i = 1 \sim n$) $\rightarrow$ Invertible

$= Ax = 0$ has only $(\vec{x} = 0)$

Let's say $\vec{x} \neq 0$, $\vec{x} = (x_1 \cdots, x_n)$

$\Rightarrow A\vec{x} = 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j = 0$ for $(i = 1 \sim n)$

$= a_{ii} x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j = 0$

some i

assume, x_i is largest $\Rightarrow |a_{ii} x_i| = \left|\sum_{j\neq i} a_{ij} x_j\right| \le \sum_{j\neq i} |a_{ij}||x_j|$

$\| $

$|a_{ii}||x_i|$ $\Rightarrow |a_{ii}| \le \sum_{j\neq i} |a_{ij}| \rightarrow$ Contradiction

- What is ~~closed~~ vector space (VS)?

→ ① 0 containing (0 matrix or 0 vector)

② Closed under addition

③ Closed under multiplication..

- Canonical transform (diagonalization)

Diagonalizable $\not\longleftrightarrow$ Singular

no correlation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$AB = T \Lambda T^{-1}$$
$$= T \Lambda T^{-1} \, T \Lambda T^{-1}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} A = T_1 \Lambda_1 T_1^{-1} \\ B = T_2 \Lambda_2 T_2^{-1} \end{cases}$$

$$\|A\|$$

$$\max_{\vec{x}} \left(\frac{\|A\vec{x}\|}{\|\vec{x}\|} \right)$$

To Review

$\boxed{\text{SVD}}$ 12/6/23

- Recall spectral decomp.

A is $n \times n$ matrix, if A is diagonalizable, then A has n lin. ind. eigenvectors $v_1 \sim v_n$ and

$$A = T \Lambda T^{-1} \iff A\vec{v_j} = \lambda_j \vec{v_j}$$

if A is symmetric, T is orthogonal matrix $(TT^{-1} = I)$.

$$\Rightarrow A = Q \Lambda Q^T \text{ where } Q \text{ is orthogonal}$$

- SVD.

A is $m \times n$ matrix, there exists orthogonal $m \times m$ and $n \times n$ matrices U and V. and a rectangular diagonal Σ w/th non-negative ~~entries~~ entries

$$\Rightarrow A = U \Sigma V^T$$

Ex: if $m > n$

non-zero ~~values~~ entries

$$m\underbrace{\begin{bmatrix} A \end{bmatrix}}_{n} = m\underbrace{\begin{bmatrix} & U & \end{bmatrix}}_{m} \overbrace{\begin{bmatrix} \diagdown \\ \Sigma \end{bmatrix}}^{n} m \overbrace{\begin{bmatrix} V^T \end{bmatrix}}^{n} n$$

$$[m \times m] \cdot [m \times n] \cdot [n \times n].$$

Let $\vec{u_1} \sim \vec{u_m}$ and $\vec{v_1}, \ldots \vec{v_n}$ are column of U and V.

since V is orthogonal, $V^TV = I = [V^T\vec{v_1} \ \cdots \ V^T\vec{v_n}] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$

$$V^T\vec{v_j} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{e_j}$$

$\rightarrow$ jth row

$-1.5 \sim 1.5$, 31 $\rightarrow \Delta x = 0.1$

$-1.5 \sim 1.5$, 11 $\rightarrow \Delta x = 0.3$

scaling : by Σ_{jj}

$\Rightarrow A v_j = U\Sigma(V^T v_j) = U\Sigma e_j = \Sigma_{jj} U e_j = \Sigma_{jj} \vec{U_j}$

$\mathbb{R}$

$\Rightarrow A \vec{V_j} = \sigma_j \vec{U_j}$ $(\sigma_j = \Sigma_{jj})$ $\longrightarrow$ right

- $\vec{U_j}$: left singular vector

  $\vec{V_j}$: right singular vector

- $A^T \vec{U_j} = V \Sigma^T U^T u_j = \sigma_j v_j$ $(\sigma_j = \Sigma_{jj})$

$\Rightarrow A^T \vec{U_j} = \sigma_j \vec{U_j} \longrightarrow$ left

$\Rightarrow$ Left singular vector A = Right singular vectors of A^T //

ML input $\rightarrow$ ML output.

- Properties

① Can always SVD $\sigma_1 > \sigma_2 \cdots \sigma_n$ (≥ 0).

Assume r non-zero singular values, $\sigma_1 \geq \cdots \geq \sigma_r > 0$

$\sigma_{r+1} = \cdots = \sigma_n = 0.$ since $Av_j = \sigma_j v_j$

(1) $u_1 \sim u_r \in col(A)$

(2). $v_{r+1} \cdots v_n \in N(A)$ e.g., $Av_{r+1} = \sigma_{r+1} v_{r+1} = 0$

Thm. In fact,

- $\vec{u_1}, \dots, \vec{u_r}$ is an orthonormal basis of $col(A)$.

- $\vec{v_{r+1}}, \dots, \vec{v_n}$ is an orthonormal basis of $N(A)$.

pf.) $\vec{u_1} \sim \vec{u_r}$ are lin. ind. so $rk(A) \geq r$

Similarly, $\vec{v_{r+1}} \sim \vec{v_n}$ are lin. ind $\Rightarrow N(A) \geq n-r$

By Rank-nullity theorem, $rk(A) + \dim(N(A)) = n$.

$$\Rightarrow rk(A) \leq n - (n-r) = r$$

$$\Rightarrow rk(A) = r$$

$rk(A) = \#$ of non-zero singular values.

→ How to find SVD decomp.?

Also

$\vec{v_1} \ \dots \ \vec{v_r} \in col(A^T)$

$\vec{u_{r+1}}, \dots, \vec{u_n} \in N(A^T)$

$$A = \begin{bmatrix} \underbrace{col(A)}_{r} & \Big| & N(A^T) \end{bmatrix} \begin{bmatrix} \ddots & \\ & \end{bmatrix} \begin{bmatrix} \underbrace{col(A^T)}_{n-r} & \Big| & \underbrace{\quad}_{r} \end{bmatrix}^T$$

- $A^TA = (V\Sigma^T U^T)(U\Sigma V^T) = V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^T$

So, spectral decomposition of A^TA is SVD

↳ Why is it guaranteed ? ⇒ A^TA is symmetric

↓

It always have spect. decomp.

How to batch ↙ $X = \{2, 3, 4\}$

$I(2)$ $I(3)$ $I(4)$ 를 예측하면, (AW)

$K(2\to3)$ $K(3\to2)$ $K(4\to3)$

~~$K(2\to1)$~~ $K(3\to4)$ ~~$K(4\to5)$~~

only $n'(3)$ 만 가능.

$I(2) \sim I(7)$ prediction (AW)

$K(3\to2)$ $K(3\to4)$, . . . $K(6\to5)$ $K(6\to7)$

$n'(3) \sim n'(6)$

Q) Why are eig vals of A^TA non-negative

$A^TAv = \lambda v \quad \|Av\|^2 = v^TA^TAv = \lambda \cdot v^Tv = \lambda\|v\|^2$

$\Rightarrow \lambda = \dfrac{\|Av\|^2}{\|v\|^2} \geq 0$

Similarly, $AA^T = U\Sigma^2 U^T$ → eig vals of AA^T is left singular

If A is ill-conditioned,

AA^T, A^TA diverges or becomes 'ill' → Not good in practice

∵ condition number of AA^T, A^TA is
proportional to (square) (cond A)

→ How to resolve this? → Bi-diagonalization.

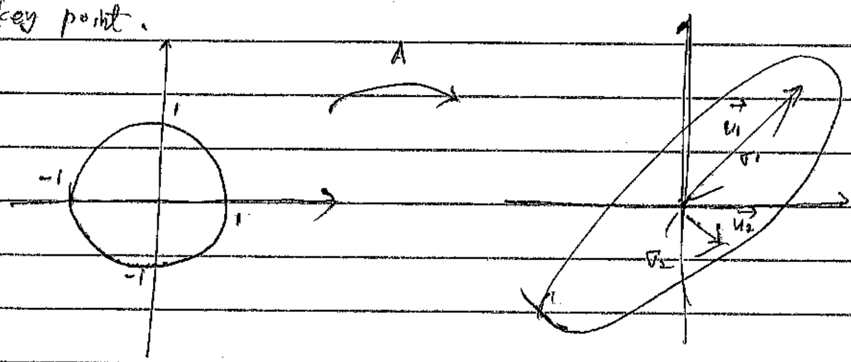
<Geometric descriptions>.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ How is unit circle transformed by A.

→ We want to show. $\vec{z} = A\vec{x}$

If A is non-singular, $\vec{x} = A^{-1}z$ → impose $\|\vec{x}\| = 1$.

key point.



stretched in directions.

1) Left singular vectors : semi-axis of resulting ellipse.
2) Singular values : are length of those axis.

Application. → Compress image → only keep vectors largest singular values
+ Capture variance in data → build recommendation systems.