ESRF   01/08/2026.

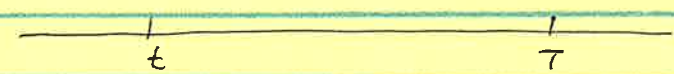
- Risk neutral pricing

  X_t : price (risky asset) $\rightarrow$ Geometric Brownian Model (GBM)

Return : $dX_t / X_t = \mu \cdot dt + \sigma \, dW_t$: return.

Problem : price a contingent contract (claim)

Ex) European call option. (ECO)

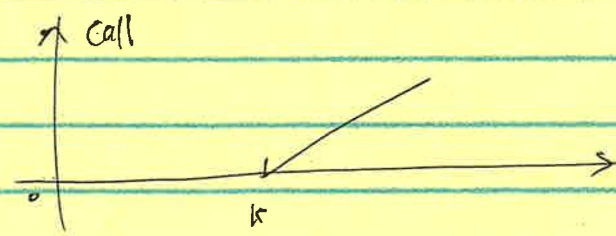


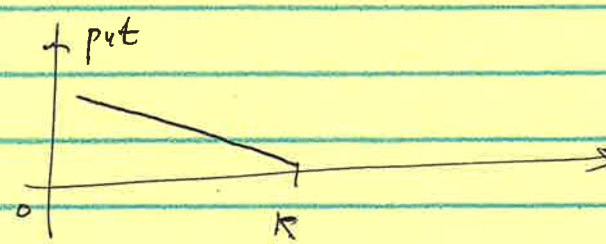
Contract pays $(X_T - k)^+ = \begin{cases} X_T - k & (X_T > k) \\ 0 & (X_T \le k) \end{cases}$ $\rightarrow$ (don't lose $x_t \ngtr X_T$)

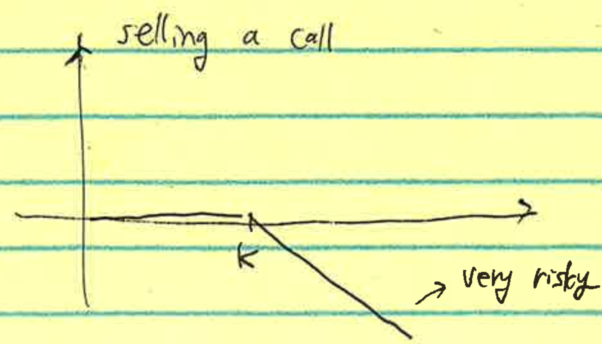
What is price of contract today at time 't'?

Also, European put option : payoff $(k - X_T)^+$: insurance.

You can use arbitrary $f(X_T)$







- How to price ?

key idea : The writer (seller) of option doesn't want to participate in any risk. (market risk)

→ The hedging portfolio ($\sim$1970).

$$\Pi_t = \underset{\substack{\downarrow \\ \text{amount of stock}}}{\Delta_t X_t} - \underset{\substack{\downarrow \\ \text{price of derivative.}}}{f(t, X_t)}$$

Δt, $f(t, X_t)$ determined.

· self financing condition (no new money).

$$d\Pi_t = \Delta_t \, dX_t - df(t, X_t)$$

$$\rightarrow d\Pi_t = \underbrace{r}_{\text{risk free interest rate}} \Pi_t \quad : \left(\begin{array}{l} \text{Force } \Pi_t \text{ to be risk free} \\ \text{by choosing } \Delta_t, f. \end{array} \right)$$

① → $$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) (dX_t)^2$$

② $$\frac{dX_t}{X_t} = \mu \cdot X_t \, dt + \sigma \, dW_t$$

$$\Rightarrow (dX_t)^2 = X_t^2 (\mu dt + \sigma dW_t)^2 \approx X_t^2 \, \sigma^2 \underbrace{dW_t^2}_{=dt}$$

$$= X_t^2 \sigma^2 dt$$

From ①, ②

$$r(\Delta_t X_t - f)dt = \Delta_t \, dX_t - df$$

$$= \Delta_t X_t(\mu dt + \sigma dW_t) - \left\{ \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} X_t(\mu dt + \sigma dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} X_t^2 \sigma^2 dt \right\}$$

$$\Rightarrow (\ \cdot\ ) dW_t + (\ \cdot\ ) dt = 0$$

$$\cancel{\sigma \Delta_t X_t} - \cancel{\sigma X_t \frac{\partial f}{\partial x}} = 0$$

$$\Rightarrow \boxed{\Delta_t = \partial f / \partial x}$$

Cancel!

$$r \Delta_t X_t - f - \cancel{\Delta_t X_t \mu} + \partial f/\partial t + \cancel{\frac{\partial f}{\partial x} X_t \cdot \mu} - \frac{1}{2}\frac{\partial^2 f}{\partial x^2} X_t^2 \sigma^2 = 0$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2} + r X_t \frac{\partial f}{\partial x} - rf = 0}$$

Black-scholes PDE.

for $\forall x \geq 0$, $0 \leq t < T$

- Black-scholes PDE

$$\frac{\partial f(t,x)}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2} + r X_t \frac{\partial f}{\partial x} - rf = 0$$

$\forall x \geq 0$, $0 \leq t < T$, $f(T,x) = h(x)$: payoff.

Why disappearance of μ important?

e.g) Temperature!

A: You cannot determine μ, but σ can be!

• Difficult to know μ because, EMH (efficient market hypothesis)
→ market (healthy) unpredictable.

$$dX_t / X_t = \mu \, dt + \sigma \, dW_t$$

$$\left\{ \begin{array}{l} \mu \to \text{difficult} \\ \sigma \to \text{how?} \end{array} \right\}$$

Let $f(t, x; T, K, \sigma, r)$ be price of call option

How does $\partial f / \partial \sigma > 0$ (volatility increase, stock price ↑)

How to estimate σ ?

Vigs (volatility index) ← calculate from liquidity moving average.

• How Π_t used to remove risk of writing option.

t, $f(t, X_t)$ by solving B.S. PDE.

→ Hedging increases volatility! → Affects the market.

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- GBM model

$dX_t/X_t = \mu\, dt + \sigma\, dW_t$: Markovian.

price of contract today : $f(t, X_t)$

pay $h(X_T)$ to the contract at T.

Note : Markovianity.

$dX_t = X_t(\mu\, dt + \sigma\, dW_t)$

$X_T - X_t = \int_t^T \mu X_s\, ds + \int_t^T \sigma X_s\, dW_s$.

$\sim \sum_{k=1}^{n-1} \sigma X_{t_{k-1}} (W_{t_k} - W_{t_{k-1}})$.

→ Forward!

- Hedging portfolio.

- $\Pi_t = \Delta_t X_t - f(t, X_t)$
- $d\Pi_t = \Delta_t\, dX_t - df(t, X_t)$
- $d\Pi_t = r \Pi_t\, dt$ (risk free).

Using Ito's formula, we find

$\Delta_t = \partial f(t, X_t)/\partial X$ (delta hedge ratio).

$f(t, X_t)$ solves the B.S. PDE.

↳ $\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} + rX\frac{\partial f}{\partial X} - rf = 0$ $(t < T)$

$f(T, x) = h(x)$

Risk neutral pricing.

Eco : $h(x) = (X-k)^+ = \begin{cases} X-k & X>k \\ 0 & X \le k \end{cases}$ Market significance.

↘ The writer of contract assures no risk.

- How RN portfolio realize risk neutrality?

$$f(t, X_t) = \Delta_t X_t - \Pi_t$$

① Sell derivative, collect $f(t, X_t)$ (Solve B.S. PDE)
② Then calculate $\frac{\partial f}{\partial x}(t, X_t) = \Delta_t$
③ Hold $\Delta_t X_t$ of the risky asset.
④ Put balance in the bank.

At time T, $\Delta_T X_T - \Pi_T = f(T, X_T) = h(X_T) = \text{payoff}$.

- Can $f(t,x)$ be written as an expectation?

Not true that $f(t, X_t) = \cancel{E\{e^{-r(T-t)} h(X_T) \mid X_t\}}$ $t<T$.
False

$\boxed{\text{Principle}}$: Every tradable instrument must be priced so that the price is martingale w.r.t. some prob. law.

$f(t, X_t)$ is a discounted amartingale.

$$E\{e^{-rt} f(t, X_t) \mid \mathcal{F}_s\} = e^{-rs} f(s, X_s) \rightarrow \text{fair game.}$$

→ Apply this to derivative pricing.
Postulate that E^* goes with X_t solving

$$dX_t / X_t = \underbrace{r\,dt}_{\text{replace } \mu \to r} + \sigma \underbrace{dW_t^*}_{\text{replace } W_t \to W_t^*}$$

$$d[e^{-rt} f(t,X_t)]$$

$$= -re^{-rt} f(t,X_t)dt + e^{-rt} f_t(t,X_t)\,dt + e^{-rt} f_x(t,X_t)\,dX_t$$

$$+ \frac{1}{2} e^{-rt} f_{xx}(t,X_t)(dX_t)^2$$

$$\left(\begin{array}{l} \text{Recall} \quad dX_t = X_t(r\,dt + \sigma\,dW_t^*) \\ \quad \Rightarrow (dX_t)^2 \approx \sigma^2 X_t^2\,dt \end{array} \right)$$

$$\Rightarrow d[e^{-rt} f(t,X_t)] = e^{-rt} \Big\{ -rf\,dt + f_t\,dt + f_x(X_t r\,dt + X_t \sigma\,dW_t^*)$$

$$+ \frac{1}{2}\sigma^2 f_{xx} X_t^2\,dt \Big\}$$

$$= e^{-rt} \Big\{ \underbrace{\{ f_t + \tfrac{1}{2}\sigma^2 X_t^2 f_{xx} + rX_t f_x - rf \}}_{\to \text{B.S. PDE!}} dt$$

$$+ e^{-rt}\sigma X_t f_x\,dW_t \Big\}$$

If 'f' solves the B.S. PDE,

$$d[e^{-rt} f(t,X_t)] = e^{-rt}\sigma X_t f_x(t,X_t)\,dW_t^*$$

$$\Rightarrow -e^{-rt} f(t,X_t) + e^{-rT} f(T,X_T) = \underbrace{\int_t^T e^{-rS}\sigma X_S f_x(S,X_S)\,dW_S^*}_{E(*)=0} {}^{*}$$

$$\Rightarrow E\{ e^{-rT} f(T,X_T) \mid \mathcal{F}_s \} - e^{-rt} f(t,X_t) = 0$$

$$\therefore \underline{E\{ e^{-rT} f(T,X_T) \mid \mathcal{F}_s \} = e^{-rt} f(t,X_t)}$$

with E^*

E^*: equivalent martingale measure

$\therefore$ Demanding $e^{-rt} f(t,X_t)$ be a martingale immediately means $f(t,X_t)$ must solve the B.S. PDE.

solution for the B.S. PDE

$$\Rightarrow \boxed{f(t,X_t) = E^*\{ e^{-r(T-t)} h(X_T) \} \quad \text{with} \quad \frac{dX_t}{X_t} = r\,dt + \sigma\,dW_t^*}$$

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- Geometric Brownian Motion (GBM).

$$dX_t / X_t = \underset{\substack{\downarrow \\ \text{mean}}}{\mu} dt + \underset{\substack{\downarrow \\ \text{volatility}}}{\sigma} dW_t.$$

Question : Price derivative contract. , $h(X_T)$: payoff

price this contract today $t<T$.

① Use hedging portfolio (BS PDE)

② Principle of no arbitrage (RNP)

$\Rightarrow$ all tradable assets must be martingale w.r.t. some $\mathcal{F}_t$

① $\Pi_t = \Delta_t X_t - f(t, X_t)$

$d\Pi_t = \Delta_t dX_t - df(t, X_t)$

$d\Pi_t = r \Pi_t dt$

$\Rightarrow \Delta_t = \frac{\partial f}{\partial x}(t, X_t)$

$$\Rightarrow \left.\begin{aligned} &\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x} - rf = 0 \\ &f(T, x) = h(x) \end{aligned}\right\}$$

B.S. PDE

② If $f(t, X_t)$ price of a derivative,

$e^{-rt} \cdot f(t, X_t)$ is martingale w.r.t. $\mathcal{F}$.

This law is $dX_t/X_t = \mu dt + \sigma dW_t^*$ (why right)

E.g.) If $h(x) = x$

Try $f(t, X_t) = a(t) \cdot x$. $\Rightarrow$ $\dot{a}x + \cancel{rxa} - \cancel{rax} = 0$ $\Rightarrow$ $a(t) = a = 1$

$f(t, X_t) = X_t$ $\Rightarrow$ $\underset{\sim\sim}{\Pi_t = 0}$. (Strategy : $f(t, X_t) = X_t$

1) Sell to client X_t

2) buy/hold X_t

3) at T, payoff X_T

Therefore, $e^{-rt}X_t$ must be martingale.

The only law is,

$$dX_t / X_t = r\,dt + \sigma\, dW_t^*$$

$$d\left(e^{-rt}X_t\right) = \underbrace{e^{-rt}\sigma X_t\, dW_t^*}$$

↘ stochastic integral, ∴ martingale.

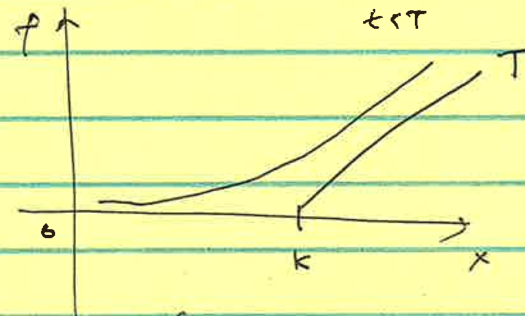
$$\Rightarrow f(t, x) = E^*\left\{ e^{-r(T-t)} h(X_T) \;\middle|\; X_t = x \right\}$$

Problem : $E\{\cdot\}$ we know dynamics but $E^*\{\cdot\}$ we do not!

Solution : E^* is dominant in the secondary option market

$$E^*\left\{ e^{-r(T-t)} (X_T - K)^+ \;\middle|\; X_t = x \right\}$$

↳ observed for many T and K.



$$-\frac{\partial^2}{\partial K^2} C(t, K, T) = e^{-r(T-t)} \underbrace{E^*\left\{ \delta(X_T - K) \;\middle|\; X_t = x \right\}}_{\parallel} \quad \left(\because \frac{\partial^2}{\partial K^2} h = -\delta\right)$$

prob density

$$p^*(t, x; T, K)$$

$p^*(t,x;T,K)$ is transition probability density $(t,x) \rightarrow (T,K)$

by Figlevski (NYU)

$$-e^{r(T-t)} \frac{\partial^2}{\partial K^2} \underbrace{C(t,K,T)}_{\downarrow} = p^*(t, x; T, K)$$

Concave ($\because p^* > 0$)

$x > k$: ~~out of~~ in the money, as $t \to T$, $d_1 \to \infty$, $N(d_1) \to 1$ (hold)

$x < k$: out of the money (refer to notes) (sell)

- Implied Volatility + Estimating Realized Volatility + stochastic volatility.

obsv.

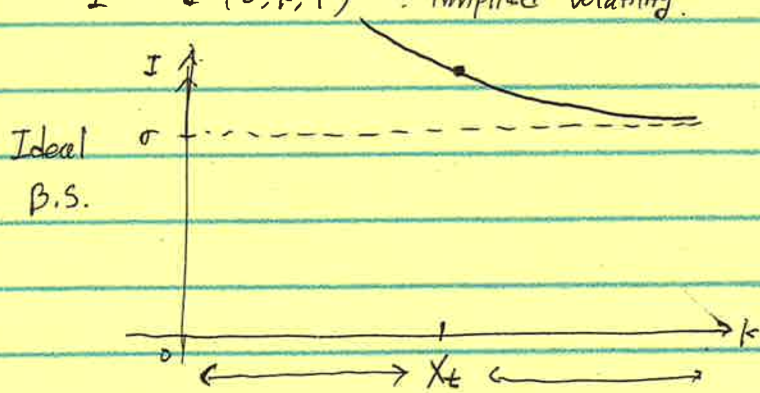
$$C_{BS}(t, k, T) = C_{BS}(t, X_t, k, T, \sigma, r)$$

We know : $\frac{\partial}{\partial \sigma} C_{BS} > 0$

For what σ do we get "equality"

$$\sigma = \sigma(t, k, T)$$

$$I = I(t, k, T)$$: implied volatility



in the money

↓

client likely to exercise

out of the money.

↓

not likely

Broker's model risk ↑ when client likely to exercise...

If $dX_t / X_t = \mu dt + \sigma dW_t$ is bad model,

→ What is a good model?

→ Can we estimate $\sigma(t, X_t)$?

→ Does IV in this model realistic? → No!

σ deterministic implies complete market ≡ perfect hedging.

↓

B.S. intact : $\begin{cases} \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t,x)\, x^2 \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x} - rf = 0, & (t < T) \\ f(T,x) = h(x) & (t = T) \end{cases}$

$$f(t, x) = f(t, x; T, k, \sigma)$$

↓ function of (t, X_t).

$$J = \sum_{i,j} \left\{ C_{obsv}(t, k_i, T_j) - f(t, X_t, T_j, k_j, \sigma) \right\}^2$$

"minimize" J w.r.t. σ

By Dupire,

$C_{BS}(t, x; T, k)$ satisfies B.S.PDE

Also, $\left.\begin{array}{l} \partial C / \partial T = \frac{1}{2}\sigma^2(T,k) \cdot k^2\, \partial^2 C / \partial k^2 - rk\, \partial C / \partial k \quad (t < T) \\ C(t,k) = (x-k)^+ \end{array}\right\}$.

$C(T,k)$ is observed → solve for σ!

$$\Rightarrow \sigma(T,k) = \left(\frac{\partial C}{\partial T} + 2\frac{\partial C}{\partial k}\right) \cdot \left(\frac{1}{2} k^2 \frac{\partial^2 C}{\partial k^2}\right)^{-1}$$

(from data). Issue: depends on 't'

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$$dX_t/X_t = \begin{Bmatrix} \mu \\ r \end{Bmatrix} dt + \sigma(t, X_t)\, d\begin{Bmatrix} W_t \\ W_t^* \end{Bmatrix}.$$

$$C(t, x; T, K) = E^*\{ e^{-r(T-t)} (X_T - K)^+ \mid X_t = x \}.$$

C satisfies the B.S.-PDE. (derived from hedging model & Martingale).

$$\left.\begin{aligned} &\partial C/\partial t + \tfrac{1}{2}\sigma^2 x^2\, \partial^2 C/\partial x^2 + r x\, \partial C/\partial x - rC = 0 \quad (t<T) \\ &C(T, x; T, K) = (x-K)^+ \end{aligned}\right\}$$

Now derive differently

D.P.-PDE (Dupire)

$$\left.\begin{aligned} &\partial C/\partial T = \tfrac{1}{2}\sigma^2(T,K) K^2\, \partial^2 C/\partial K^2 - rK\, \partial C/\partial T, \quad (T>t) \\ &C(t, x; t, K) = (x-K)^+. \end{aligned}\right\}$$

From this we can estimate,

$$\sigma^2(T,K) = \frac{\partial C/\partial T(T,K) + rK\, \partial C/\partial K(T,K)}{\frac{1}{2}K^2\, \partial^2 C/\partial K^2 (T,K)} \qquad (\text{for a fixed } (t,x))$$

False impression : σ $\boxed{\text{does}}$ depend on (t, x)

(Note p.29)

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

Ito's formula, $df(t, X_t) = f_t\, dt + f_x\, dX_t + 1/2\, f_{xx} \underbrace{(dX_t)^2}_{= \sigma^2 \cancel{(} X_t\, dt)}$

$$= \left(f_t + \tfrac{1}{2}\sigma^2 f_{xx} + \mu f_x\right) dt + \sigma f_x\, dW_t$$

$\boxed{\text{B.K.E.}}$

B.K.O.

Integrate $t \to T$, $\quad \mathcal{L}_t = \frac{1}{2}\sigma^2(t,X_t)\,\partial^2/\partial x^2 + \mu(t,X_t)\,\partial/\partial x$

$\to 0$ (if B.K.E satisfied)

$$\Rightarrow \quad f(T,X_T) - f(t,X_t) = \int_t^T \cancel{\left(\partial/\partial s + \mathcal{L}_s\right)} f(s,X_s)\,ds$$
$$+ \int_t^T \sigma(s,X_s) f_x(s,X_s)\,dW_s$$

Suppose f solves B.K.E.

$$\left(\partial/\partial t + \mathcal{L}_t\right) f(t,x) = 0 \quad, (t<T)$$
$$f(T,x) = h(x)$$

Then, $\quad h(X_T) - f(t,X_t) = \int_t^T \sigma(s,X_s) f_x(s,X_s)\,dW_s$

Take $E\{\,\cdot\,|X_t = x\}$, $= 0$ (zero mean martingale)

$$E\left\{\int_t^T \sigma f_x\, dW. \,\middle|\, X_t\right\} = 0$$

Ideal Markov case

$$\therefore \quad f(t,X_t) = E\{h(X_T)\,|\,X_t = x\}$$

Let $p(t,x,T,\xi)$: transition probability density of $X_\cdot$ @ T w.r.t ξ given $X_t = x$

$$f(t,x) = \int h(\xi)\,p(t,x,T,\xi)\,d\xi.$$

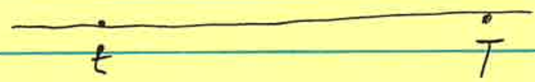
$$(\partial_t + \mathcal{L}_t) f = 0 = \int h(\xi)\left(\partial_t + \mathcal{L}_t\right) p(t,x;T,\xi)\,d\xi = 0$$

$$\therefore \left(\partial_t + \mathcal{L}_t\right) p(t,x;T,\xi) \quad (t<T)$$
$$p(T,x,T,\xi) = \delta(x-\xi)$$

$\boxed{\text{FKE}}$

Equation for p as function of (T, z).

Let $\phi(t, z)$ be test function.



$$\begin{cases} \phi(0, z) = \phi(T, z) = 0. \\ \phi(t, z) \equiv 0 \quad \text{if } |z| \gg 1 \end{cases}$$

$$d\phi(t, X_t) = (\phi_t + \mathcal{L}_t \phi)\,dt + \sigma \phi_x \, dW_t$$

Integrate)
$$\cancel{\phi(T, X_T)}^{\,0} - \cancel{\phi(t, X_t)}^{\,0} = \int_t^T (\phi_s + \mathcal{L}_s \phi)\,ds + \int_t^T \sigma \phi_x \, dW_s$$

Expectation)
$$\underbrace{0 = E\left\{ \int_t^T (\phi_s + \mathcal{L}_s \phi)\,ds \,\middle|\, X_t = x \right\}}$$

$$\Rightarrow 0 = \int dz \int_t^T ds \left(\phi_s(s, z) + \mathcal{L}_s \phi(s, z) \right) \cdot p(t, x, s, z)$$

Identity for all $\phi(t, x)$

Integrate parts)
$$0 = \int dz \cdot \int_t^T ds \; \underbrace{p(t, x; s, z)}\underbrace{\left(\frac{\partial}{\partial s} + \mathcal{L}_s \right)\phi(s, z)}$$

$$\mathcal{L}_s = \frac{1}{2}\sigma^2(s, x)\, \partial^2/\partial x^2 + \mu(s, x)\, \partial/\partial x$$

$$\Rightarrow 0 = \int dz \cdot \int_t^T ds\, \phi \left(-\partial/\partial s + \mathcal{L}_s^* \right) p$$

If p differentiable, $p(T, z)$,
$$\left.\begin{aligned} \partial p/\partial T &= \mathcal{L}_T^* p \quad (T > t) \\ p(t, z) &= \delta(x - z) \end{aligned}\right\}$$

$$\mathcal{L}_T^* = \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2(T, z)\, \cdot \right) - \frac{\partial}{\partial z}\left(\mu(T, z)\, \cdot \right)$$

- Dupire Eq. (p. 51 of notes)

$$c(T,k) = E^*\{ e^{-r(T-t)} (X_T - k)^+ \mid X_t = x \}$$

$$= e^{-r(T-t)} \int_0^\infty (\xi - k)^+ \, p(t,x;T,\xi) \, d\xi$$

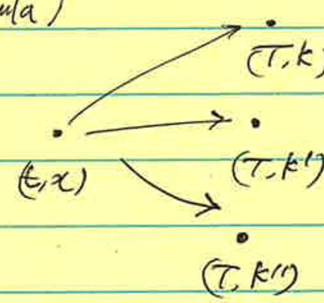
$$\partial c / \partial T =$$

$$\partial c / \partial k = - e^{-r(T-t)} \int_0^\infty \mathbb{1}(\xi > k) \, p(t,x;T,\xi) \, d\xi$$

$$\partial^2 c / \partial k^2 = + e^{-r(T-t)} \int_0^\infty \delta(\xi - k) \, p(t,x;T,\xi) \, d\xi$$

$$= e^{-r(T-t)} \, p(t,x;T,k) \quad (\text{B.L. formula})$$

Transition probability (valuable).



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- $\boxed{dX_t/X_t = \mu dt + \sigma dW_t}$ →

→ Dupire Equation / Complete it!
→ Local volatility (LV)
→ Stochastic Volatility

$$dX_t/X_t = r\,dt + \underbrace{\sigma(t, X_t)}_{\text{unknown.}} dW_t^*$$

"objective : Determine LV from options data"

B.S for option prices (ECo)

$$C(t, X_t; T, k) = E^*\{ e^{-r(T-t)} \cdot (X_T - k)^+ | X_t \}$$

The B.S. PDE for $C(t, x, T, k)$ as (t,x) function,

$$\partial C/\partial t + \tfrac{1}{2}\sigma^2(t,x)\, x^2\, \partial^2 C/\partial x^2 + r x\, \partial C/\partial x - r C = 0 \qquad (t<T)$$

$$C(T, X, T, k) = (x-k)^+ \qquad (t=T)$$

Now, derive Dupire Eq. $C(T,k)$

$$\partial C/\partial T = 1/2\, \sigma^2(T,k)\, k^2\, \partial^2 C/\partial k^2 - rk\, \partial C/\partial k \qquad (T>t)$$

$$C(t, x, T, k) = (x-k)^+ \qquad @T$$

Formally we can obtain

$$\underbrace{\sigma^2(T,k) = \frac{\partial C/\partial T\,(T,k) + k\, \partial C/\partial k\,(T,k)}{\frac{1}{2} k^2\, \partial^2 C/\partial k^2\,(T,k)}}_{\text{computable from 'data'}}$$

X_t is a process,

$p(t, x; T, k)$: transition probability density of Markov process.

for X $(t, x) \to (T, k)$ and $T > t$

↓ Satisfies BKE, FKE.

BKE: $\partial p/\partial t + \mathcal{L}_t p = 0 \quad (t < T)$

$p(T, x; T, k) = \delta(x-k)$

FKE: $\partial p/\partial T = \mathcal{L}_T^* p \quad (T > t)$

$p(t, x; t, k) = \delta(x-k)$

$\mathcal{L}_t = \frac{1}{2}\sigma^2(t, x)\, x\, \partial^2/\partial x^2(\cdot) + rx\, \partial/\partial x(\cdot)$

(p. 29 of notes)

$\mathcal{L}_T^* = \frac{1}{2}\frac{\partial^2}{\partial k^2}\left(\sigma^2(T, k)k^2 \cdot\right) - \partial/\partial k\left(rk \cdot\right)$

Using transition density,

$$C(t, x; T, k) = E^*\{e^{-r(T-t)}(x-k)^+ | X_t\}$$

$$= e^{-r(T-t)}\int_0^\infty (\xi-k)^+ p(t, x; T, \xi)\, d\xi \quad \text{------ } (*)$$

Calculate $\partial C/\partial T$, $\partial C/\partial k$, $\partial^2 C/\partial k^2$ and find relation.

$$\partial C/\partial k = -e^{-r(T-t)}\int_0^\infty \mathbb{1}\{\xi > k\}\, p(t, x; T, \xi)\, d\xi$$

$$\partial^2 C/\partial k^2 = e^{-r(T-t)}\int_0^\infty \delta(\xi - k)\, p(t, x; T, \xi)\, d\xi = e^{-r(T-t)} p(t, x; T, k)$$

(B.L. formula)

$$\partial C/\partial T = e^{-r(T-t)}\int_0^\infty (\xi-k)^+ \underbrace{\frac{\partial p}{\partial T}(t, x; T, \xi)}_{\text{FKE}}\, d\xi - re^{-r(T-t)}\int_0^\infty (\xi-k)^+ p(t, x; T, \xi)\, d\xi$$

$$\Rightarrow \partial C/\partial T = e^{-r(T-t)} \cdot \int_0^\infty (z-k)^+ \mathcal{L}_T^* \, p(t,x;T,z)\, dz$$

$$- r e^{-r(T-t)} \cdot \int_0^\infty (z-k)^+ \cdot p(t,x;T,z)\, dz$$

Integrate by parts.

$$= e^{-r(T-t)} \int_0^\infty p(t,x;T,z) \cdot \Big[\tfrac{1}{2}\sigma^2(T,z)\, z^2 \, \delta(z-k)$$

$$- r z \, \mathbb{1}(z>k) \Big]\, dz$$

$$- r e^{-r(T-t)} \int_0^\infty (z-k)^+ p \cdot dz$$

$$= \tfrac{1}{2}\sigma^2(T,k)\, k^2 \, e^{-r(T-t)} \, p(t,x;T,k)$$

$$+ r e^{-r(T-t)} \int_0^\infty z \, \mathbb{1}(z>k)\, p$$

$$- r e^{-r(T-t)} \int_0^\infty (z-k)^+ p$$

$$= \tfrac{1}{2}\sigma^2(T,k) \cdot k^2 \, \partial^2 C/\partial k^2 + \underbrace{r k e^{-r(T-t)} \int_0^\infty \mathbb{1}(z>k)\, p \, dz}_{-rk\,\partial C/\partial k}$$

$$\Rightarrow \underline{\partial C/\partial T = \tfrac{1}{2}\sigma^2(T,k)\, k^2 \, \partial^2 C/\partial k^2 - rk\, \partial C/\partial k} \quad (p.\ 52)$$

1) local vol interesting — Comple market → hedge exactly.
2) local vol to be avoided — depends strongly on (t,x)

Silent premise: σ depends only on (t,x) ① → (T,k) ②

- Stochastic Volatility. ((1980 − 1990) by 'Hull − White'

$$dX_t = \mu_t X_t \, dt + \sigma_t X_t \, dW_t^{(0)}$$

μ_t, σ_t random, possibly depend on $W_t^{(0)}$

one-factor model : $\sigma_t = f(Y_t)$

not observable.

$$dY_t = \alpha(Y_t)\,dt + \beta(Y_t)\,dW_t^{(1)}$$

$W_t^{(0)}$, $W_t^{(1)}$ are B.M.s $\quad dW_t^{(0)}\,dW_t^{(1)} = \rho\,dt \quad$ (recall $(dW_t)^2 = dt$)

(If $W_t^{(0)}$, $W_t^{(1)}$ is independent, $\rho = 0$)

$\rho > 0$ implies correlation $(-1 \le \rho \le 1)$.

$$\operatorname{corr}(d\sigma_t, dX_t) = \frac{\operatorname{cov}(d\sigma_t, dX_t)}{\sqrt{\operatorname{var}(d\sigma_t)\cdot\operatorname{var}(dX_t)}}$$

$$d\sigma_t = df(Y_t) = f'(Y_t)\,dY_t + \frac{1}{2} f''(Y_t)\underbrace{(dY_t)^2}_{dt}$$

$$dX_t = \mu_t X_t \, dt + \sigma_t X_t \, dW_t^{(0)}$$

$$d\sigma_t = f'(\alpha\,dt + \beta\,dW_t^{(1)}) + \frac{1}{2} f'' \beta^2 \, dt$$

$\therefore \operatorname{cov}(d\sigma_t, dX_t) = f(Y_t)\cdot X_t\, f'(Y_t)\cdot \beta(Y_t)\,\rho\,dt$

$\therefore (d\sigma_t)^2 = (f')^2 \beta^2 \, dt$, $(dX_t)^2 = \sigma_t^2 X_t^2 \, dt$

$\therefore \ \mathrm{Corr}(d\sigma_t, dx_t) = \rho$

Interpretation : correlation btw. $\begin{pmatrix} \text{volatility fluctuations} \\ \text{price fluctuations} \end{pmatrix}$

(volatility $\propto$ volume.)

01/27/2026

- Stochastic Volatility.

- Risky asset price model

$$dX_t = \mu_t \bar{X}_t \, dt + \sigma_t \bar{X}_t \cdot dW_t^{(0)}$$

μ_t, σ_t are random.

Model σ_t (one-factor) : Factor process Y_t , $\sigma_t = f(Y_t)$

$$dY_t = \alpha(Y_t)dt + \beta(Y_t)\,dW_t^{(1)}$$

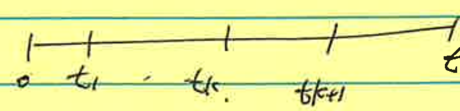
α, β are given.

Assume, $dW_t^{(0)} \cdot dW_t^{(1)} = \rho \, dt$

ρ: Correlation of fluctuations. $(-1 \le \rho \le 1)$

Ex) $(dW_t)^2$: infinitesimal Q.V.

$$QV(W_t) = \lim_{\max_k (t_{k+1} - t_k) = 0} \sum_{k=0}^{N-1} (W_{t_{k+1}} - W_{t_k})^2 = t$$



$$\Rightarrow (dW_t)^2 = d\,QV(W_t) = dt.$$

$$dW_t^{(0)} dW_t^{(1)} = 0 \quad \text{(independent)}$$

$$= dt \quad (\text{correlated}, \ \rho = 1)$$

- Complete SV model (p.56)

$$dX_t = \mu(X_t)\,X_t\,dt + \sigma_t\,X_t\,dW_t^{(0)}$$

$$\sigma_t = f(Y_t) \qquad (f : \text{unknown})$$

$$dY_t = \alpha(Y_t)\,dt + \beta(Y_t)\,dW_t^{(1)}$$

$$W_t^{(1)} = \underbrace{\rho\,W_t^{(0)} + \sqrt{1-\rho^2}\;W_t^{\perp}}_{\equiv\text{"projection"}} \qquad (W_t^{(0)} \text{ and } W_t^{\perp} \text{ independent B.M.})$$

ρ : measure of correlation btw volatility & price

Ex) U.S. equity $\rho < 0$ $(\rho \cong -0.4)$

Currency conversion $\rho = 0$

Popular Models (~1990) — p.54

- Hallen-White $\rho = 0$, $f = \sqrt{Y}$, $Y \sim$ log-normal process.
- Scott $\rho = 0$, $f = e^{Y}$ $Y \sim$ Ornstein - Uhlenbeck.
- Stein-Stein. $\rho = 0$, $f = |Y|$ $Y \sim$ OU
- Bull - Roma $\rho = 0$, $f = \sqrt{Y}$ $Y \sim$ CIR.
- ✻ Heston $\rho \neq 0$, $f = \sqrt{Y}$ $Y \sim$ CIR

  ↘ Exact solution for option prices. (real data not matching ...)

- Bergomi (2007) ~

- Background

Fact : Prices (generally) not mean reverting
Volatility are mean reverting.

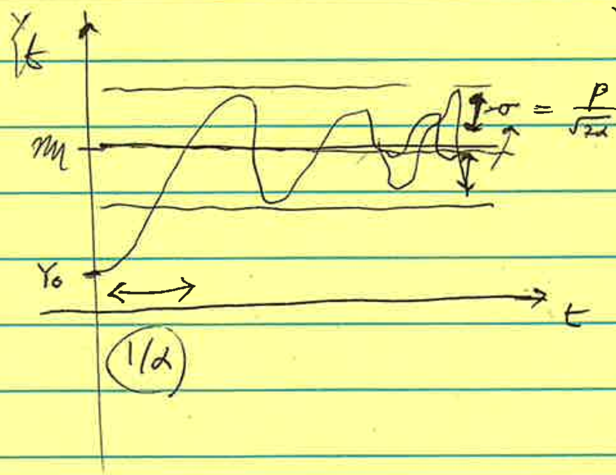
Main Model for mean reversion : OU process.

OU: $dY_t = \alpha(m - Y_t)\,dt + \beta\, dW_t^{(1)}$

→ Solvable (variation of constants)

$$Y_t = m + e^{-\alpha t}(Y_0 - m) + \beta \int_0^t e^{-\alpha(t-s)}\, dW_s^{(1)}$$

$Y_0 = Y_0$, $E[Y_t] = m + e^{-\alpha t}(Y_0 - m)$



$$\mathrm{Var}[Y_t] = E\left[\left(\beta \int_0^t e^{-\alpha(t-s)}\, dW_s^{(1)}\right)^2\right]$$

$$= \beta^2 \int_0^t e^{-2\alpha(t-s)}\, ds \quad \text{(Ito isometry)}$$

$$= \frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)$$

$$\sigma(Y_\infty) = \beta / \sqrt{2\alpha}$$

α : rate of mean reversion [$1/\alpha$: mean reverting time.]

m : long term mean.

→ Time to enter eq. band.

$\beta/\sqrt{2\alpha}$: " std.

1) $\mathrm{cov}(Y_t, Y_{t+h}) \sim \beta^2/(2\alpha)\, e^{-2\alpha h}$?

2) Time to return to $Y_t = m$ from $Y_0 = m$. and exit band $\frac{3\beta}{\sqrt{2\alpha}}$ before t ?

Parameters of OU : α, m, β.

Ex) S&P 500 (Volatility) $m \sim 15\%$

$1/\alpha \sim$ 1 week − 60 days.

$\beta \sim 10\%$

$\rho \sim -0.4\%$

- CIR process

$$dY_t = \alpha(m - Y_t)\,dt + \beta\sqrt{Y_t}\,dW_t^{(1)}$$

$(Y_t > 0)$

- How to price options in market with S.V. (H-W method)

Hedging portfolio.

$$\Pi_t = \underbrace{P^{(1)}(t, X_t, Y_t)}_{\text{price of derivative}} - \overset{\text{hedge ratio}}{A_t X_t - \Sigma_t} \cdot \underbrace{P^{(2)}(t, X_t, Y_t)}_{\text{price of derivative @ longer maturity}}$$

Self-financing

$$\begin{cases} d\Pi_t = dP^{(1)} - A_t\,dX_t - \Sigma_t\,dP^{(2)} & (A_t, \Sigma_t \text{ no derivative}) \\ d\Pi_t = r\Pi_t\,dt & \end{cases}$$

Problem : Determine $P^{(1)}, P^{(2)}, A_t, \Sigma_t$

Outline
1) $d\Pi_t \to$ Ito's formula $\to dP^{(1)}, dP^{(2)}$
2) Use only $W_t^{(0)}$ and $W_t^{\perp}$ (indep.) $W_t^{(1)} = \rho W_t^{(0)} + \sqrt{1-\rho^2}\,W_t^{\perp}$
3) Cancel $dW_t^{\perp}$ coeffs $\to \Sigma_t = \partial P^{(1)}/\partial y \,/\, \partial P^{(2)}/\partial y$
4) Cancel dW_t^{0} coeffs $\to A_t = \partial P^{(1)}/\partial x - \Sigma_t \cdot \partial P^{(2)}/\partial x$
5) dt coeffs.
6)

Previews : $\mathcal{L}_{(x,y)}$

$$\rightarrow \left(\partial p^{(1)}/\partial y\right)^{-1} \cdot \mathcal{L}\, p^{(1)} = \left(\partial p^{(2)}/\partial y\right)^{-1} \mathcal{L}\, p^{(2)}$$

(for $\forall\ t, x, y$)

doesn't depend on specific derivative.
"Characteristic" of secondary market.

$\Rightarrow$ There exists a function λ (separation constant)
s.t. $\mathcal{L} p \cdot \left(\partial p/\partial y\right)^{-1} = \lambda$

unknown.

$\therefore$ Generalized B.S. PDE

$$\partial p/\partial t + \frac{1}{2} f^2(y)\, X^2\, \partial^2 f/\partial x^2 + \rho\, \beta(y)\, X\, f(y)\, \partial^2 p/\partial x \partial y$$

$$+ \frac{1}{2} \beta(y)^2\, \partial^2 f/\partial p^2 + r\left(x\, \partial p/\partial x - p\right) \qquad (\sim\ :\ B.S\ PDE)$$

$$+ \left(\alpha(y) - \beta(y)\right) A \cdot \partial p/\partial y = 0$$

"Volatility risk premium"
Market $\rightarrow$ cannot determine
2^{nd} Market $\rightarrow$ determines this!

2026/02/03.

- $Y_t \sim N(m, \beta^2/2\alpha)$ $t \to \infty$ Y_t : OU process.

$f(Y_t) = e^{\gamma Y_t}$ → Spiky behavior (γ)

$\rho = -0.4$ (U.S. stock)

If $\rho = 0$, price / volatility fluctuations decoupled. → simplified.

$$\Pi_t = \underbrace{P^{(1)}(t, X_t, Y_t)}_{\text{target derivative price}} - A_t X_t - \underbrace{I_t}_{\text{hedge ratio.}} P^{(2)}(t, X_t, Y_t)$$

$P^{(2)}$: price of other derivative, $T_2 > T_1$ ($T_1 = T$, maturity).

Risk free portfolio : $d\Pi_t = r\Pi_t \, dt$. $\longrightarrow A_t, I_t, P^{(1)}, P^{(2)}$.

$$d\Pi_t = dP^{(1)} - A_t \, dX_t - I_t \, dP^{(2)}$$

→ $$dP^{(1)}(t, X_t, Y_t) = \partial P/\partial t \, dt + \mathcal{L}_{(X,Y)} P dt + (\quad) dW_t^{(0)} + (\quad) dW_t^{(1)}$$

$$dP = \partial P/\partial t \, dt + \partial P/\partial X \, dX + \partial P/\partial Y \, dY$$

$$+ \frac{1}{2}\left[\frac{\partial^2 P}{\partial X^2}(dX)^2 + 2\frac{\partial^2 P}{\partial X \partial Y} dX_t dY_t + \frac{\partial^2 P}{\partial Y^2}(dY_t)^2\right]$$

$(dY_t)^2 = \beta^2(Y_t) \, dt$

$dY_t dX_t = \sigma_t X_t \beta(Y_t) \rho \, dt$

$\mathcal{L}(XY) = \sim\sim$.

$$\therefore\ dP = \frac{\partial P}{\partial t}dt + \mathcal{L}_{X,Y} P\, dt$$

$$+ \left\{ f(y)\, x\, \partial P/\partial x + \beta(y)\rho \frac{\partial P}{\partial y} \right\} dW_t^{(0)}$$

$$+ \left(\sqrt{1-\rho^2}\, \beta(y) \frac{\partial P}{\partial y} \right) dW_t^{\perp}$$

$$d\Pi_t = \Big[\underbrace{\left(\frac{\partial}{\partial t} + \mathcal{L}_{X,Y}\right) P_1^{(1)} - A_t \mu X_t - \mathcal{I}_t \left(\frac{\partial}{\partial t} + \mathcal{L}_{X,Y}\right) P^{(2)}}_{=0} \Big] dt$$

$$+ \Big[\underbrace{\qquad\qquad\qquad}_{=0} \Big] dW_t^{(0)}$$

$$+ \left\{ \underbrace{\sqrt{1-\rho^2}\, \beta(x) \left[\frac{\partial P^{(1)}}{\partial y} - \mathcal{I}_t \frac{\partial P^{(2)}}{\partial y} \right]}_{=0} \right\} dW_t^{\perp} = 0$$

$$\Rightarrow \left.\begin{aligned} \mathcal{I}_t &= \partial P^{(1)}/\partial y \Big/ \partial P^{(2)}/\partial y \\ A_t &= \partial P^{(1)}/\partial x - \mathcal{I}_t\, \partial P^{(2)}/\partial x. \end{aligned}\right\}$$

Define, $\tilde{\mathcal{L}} = \partial/\partial t + \mathcal{L}(x,y) - (\mu(y) - r)x\, \partial/\partial x - r$

$$\Rightarrow \boxed{\frac{\tilde{\mathcal{L}} P^{(1)}}{\partial P^{(1)}/\partial y} = \frac{\tilde{\mathcal{L}} P^{(2)}}{\partial P^{(2)}/\partial y}} \ (*)$$

$\rightarrow$ This ratio is universal function of any derivative price regardless of payoff @ maturity.

Define ratio ⊛ as,

$$\Lambda(t,x,y) = \rho\,\frac{f(y)-r}{f(y)} + \gamma(t,x,y)\sqrt{1-\rho^2}$$

Then, the PDE satisfied by any P is (HW generalization)
(Hull-white)

$$\Rightarrow \partial P/\partial t + \frac{1}{2}\cdot f^2(y)\cdot x^2\,\partial^2 P/\partial x^2 + \rho\cdot\beta(Y)\cdot x\,f(y)\cdot\frac{\partial^2 P}{\partial x\,\partial Y} + \frac{1}{2}\beta^2(y)\cdot\frac{\partial^2 P}{\partial y^2}$$

$$+ r\left(x\frac{\partial P}{\partial x} - P\right) + \left\{\alpha(y) - \boxed{\beta(y)\Lambda(t,x,y)}\right\}\frac{\partial P}{\partial y} = 0$$

~ : Black-scholes term.

□ : only secondary market

In principle, $\alpha(y)$ & $\beta(y)$ can be estimated from market.

But not $\Lambda(t,x,y)$.

Why $\rho=0$ makes simple ? → Separation of Variables

$P = X(x)\,Y(y)\,T(t)$ → independent terms.

$$dX_t = r\,X_t\,dt + f(Y_t)\,X_t\,dW_t^{(0)*}$$
$$dY_t = (\alpha(Y_t) - \beta(Y_t)\Lambda_t)\,dt + \beta\,dW_t^{(1)*}$$

$$W_t^{(1)*} = \rho\,W_t^{(0)*} + \sqrt{1-\rho^2}\,W_t^{\perp *}$$

Then, $P(t,x,y) = E^*\left\{e^{-r(T-t)}h(X_T)\,\middle|\,X_t = x, Y_t = y\right\}$

From Risk Neutral (Λ indep of X)

$$P(t,x,y) = E^*\{E^*\{e^{-r(T-t)}h(X_T) \mid \underbrace{\mathcal{F}_T^Y}_{\text{all info about } Y \text{ process.}} \} \mid X_t, Y_t\}$$

$R.N.$

(p. 55, 56, 57, 58 59)

Intro. HW Universality gen B.S.

2026/02/06

- Interest rates
  - observable
  - Not tradable.
- Bond prices
  - Tradable, not observable.

$\Rightarrow$ Model pricing of bonds based on short term model of int. rates

r_t = short term rate.

↳ without trend, mean reverting

↳ Has volatility less than that of stocks.

Model r_t by OU process.

$$dr_t = \underbrace{a}_{\text{rate of mean reversion}}(\underbrace{r_\infty}_{\text{long term average}} - r_t)\,dt + \underbrace{\sigma}_{\text{volatility}} dW_t.$$

$1/a \sim 2$ months.

$\sigma \sim 2\text{-}4\%$

$r_\infty = 4\%$ (20 years)

$\Lambda(t,T)$: price of $\underbrace{\text{zero coupon}}$ bond today maturing @ $T > t$.

Bond pays \$1 at maturity

$\Lambda(t,T)$ is the value of \$1 today

If r = constant, $\Lambda(t,T) = e^{-r(T-t)} \cdot 1$

If r = known, $r(t)$, $\Lambda(t,T) = e^{\left(-\int_t^T r_s\,ds\right)}$

$$(\dot{x,p}) \quad \dot{x} = p$$
$$\dot{p} = -\nabla U - \gamma p + \sqrt{2\gamma^{-1}\beta^{-1}}\, W_t \ ?.$$

By r_t is a process.

~~$\Lambda(t,T)$~~ $\exp\left(-\int_0^t r_s ds\right)\Lambda(t,T)$ should be Martingale. to some law.

Law is equivalent to the law. of observed $r_t \equiv$ OU

$\Rightarrow$ σ must be same for both. (

$$\Rightarrow \; dr_t = a(r^* - r_t)dt + \sigma dW_t^* \quad (\text{what is } r^* ?)$$

$$e^{-\int_0^t r_s ds}\Lambda(t,T) = E^*\left\{ e^{-\int_0^T r_s ds}\,\cancel{\Lambda(T,T)}^{\$1} \,\middle|\, \mathcal{F}_t \right\} \quad (\text{Martingale})$$

all info for until t.

r is observed, not traded.

$$\Lambda(t,T) = E^*\left\{ \underbrace{e^{-\int_t^T r_s ds}}_{} \cdot 1 \,\middle|\, \mathcal{F}_t \right\}$$

exp of Integral of OU $\rightarrow$ Gaussian!

$$r_t \text{ is OU} \Rightarrow r_t = r^* + e^{-\alpha t}(r_0 - r^*) + \sigma\int_0^t e^{-\alpha(t-s)}dW_s$$

$$\Rightarrow r_t \sim \mathcal{N}\left(r^* + e^{-\alpha t}(r_0 - r^*),\ \frac{\sigma^2}{2\alpha}\cdot(1 - e^{-2\alpha t})\right)$$

$$\Rightarrow \Lambda(t,T) = E^*\left\{ e^{-\int_t^T r_s ds} \,\middle|\, r_t \right\} = \underline{P(t, r_t; T)}$$

$P(t, r_t; T)$ satisfies PDE.

Notice $e^{-\int_0^t r_s ds} P(t, r_t; T)$ is Martingale.

$$d\left(e^{-\int_0^t r_s ds} \cdot P(t, r_t; T) \right) = (\quad) dt + (\qquad) dW_t^*$$

$$= -r_t \, e^{-\int_0^t r_s ds} P \, dt$$

$$+ e^{-\int_0^t r_s ds} \left[\frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} dr_t + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr_t)^2 \right]$$

$$= e^{-\int_0^t r_s ds} \underbrace{\left[\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + a(r^* - r)\frac{\partial P}{\partial r} - r_t P \right]}_{=0} dt + (\quad) dW_t^*$$

EPt.

$\therefore P(r_t, t)$ solves,

$$\Rightarrow \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + a(r^* - r)\frac{\partial P}{\partial r} - r_t P = 0 \qquad t < T$$

$$P(r_t, t = T) = 1 \qquad t = T$$

→ PDE for bond price. ✓ (r^* we don't know)

Recall $\Lambda(t, T) = P(t, r_t; T)$.

Define $R(t, \tau) = -\tau^{-1} \log \Lambda(t, t+\tau)$

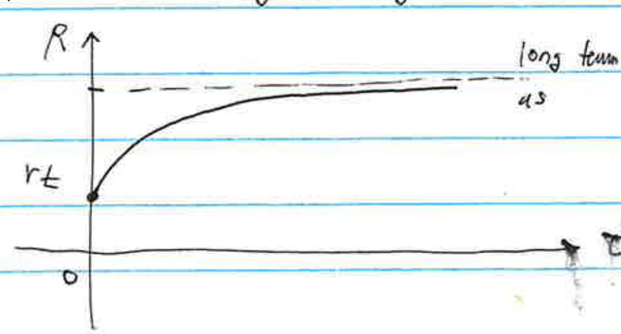
where $\Lambda(t, t+\tau) = e^{-\tau R(t,\tau)}$

τ: time to maturity

"Since $R(t,\tau)$ observed, then R_∞ is observed $\Rightarrow \left(R^\infty + \frac{\sigma^2}{2a^2} = r^*\right)$"

$R(t,\tau)$: instantaneous interest rate today maturing after τ.

"is observed"



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02/10/2026.

Read: "convertible bonds note"

$dX_t / X_t = \mu \, dt + \sigma \, dW_t \qquad X_t \leftarrow$ risky asset

$W(t)$ = value of portfolio $= a(t) X(t) + b(t) e^{rt}$

$\rightarrow$ How to select (a, b) to optimize workflow.

+

Portfolio is self-financing

$$dW(t) = \underbrace{a(t)\, dX(t)}_{\text{stock}} + \underbrace{b(t)\, r e^{rt}}_{\text{bank}} dt$$

Let $u(t)$: in stock (fraction)

$1-u(t)$: in bank (fraction)

$$\Rightarrow \left.\begin{aligned} a(t)\, X(t) &= u(t) W(t) \\ b(t)\, e^{rt} &= (1-u(t)) W(t). \end{aligned}\right\} \Rightarrow dW(t) = \frac{u(t) W(t)}{X(t)} dX(t) + \frac{(1-u(t)) W(t)}{\cancel{e^{rt}}} r \cancel{e^{rt}} dt$$

$$\Rightarrow dW(t) = u(t) W(t) \left(\mu \, dt + \sigma \, dW(t) \right) + (1-u(t)) W(t) \cdot r \, dt$$

✓

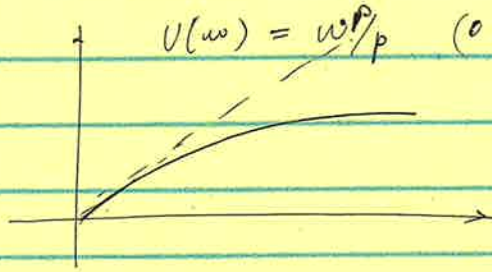
$$\Rightarrow \frac{dW(t)}{W(t)} = \left[r + (\mu - r) u(t) \right] dt + \underbrace{\sigma \, u(t)}_{\text{Volatility is weighted.}} dW(t)$$

Q) How to measure value of wealth?

[illegible])

- Utility : $U(x)$ is a utility if increasing & concave.

Ex.) $U(w) = w^p/p \quad (0 < p < 1)$



Ex) $U(w) = \log w$ → most intuitive (e.g. gamblers)

$$V(t, w) = \sup_{u(\cdot)} E\left\{ \overset{\text{Utility.}}{U}(W(T)) \,\middle|\, W(0) = w \right\}$$

$(0 \le t \le T)$

↓ value function of portfolio. (RL?)

→ show $V(t, w)$ satisfies Hamilton – Jacobi – Bellman equation.

$$\Rightarrow \partial V/\partial t + \sup_u \left\{ \frac{1}{2} \sigma^2 u^2 w^2 \frac{\partial^2 V}{\partial w^2} + \left(r + (\mu - r) u\right) w \frac{\partial V}{\partial w} \right\} = 0$$

$(t < T)$

$$V(T, w) = U(w)$$

Solution for $U(w) = w^p/p$ → optimal $u^* = \dfrac{\mu - r}{\sigma^2 (1-p)}$ = constant.

↓ $V(t, w) =$ ~~~~~.

$$= \frac{w^p}{p} \exp\left(\underbrace{r + \frac{(\mu - r)^2}{2\sigma^2 (1-p)}}\right) p (T - t)$$

↓ $p\left(r + \dfrac{(\mu - r)^2}{2\sigma^2 (1-p)} \right)$: optimal growth rate.

$\dfrac{\mu - r}{\sigma^2 (1-p)}$: opt. allocation ratio.

$\dfrac{\mu - r}{\sigma}$: $\boxed{\text{"sharpe" ratio.}}$

$(\sim 1/\sqrt{\text{time}})$

- Fundamental theory of rebalancing
  $\Rightarrow$ "Buy low sell high".

- Log utility.. (maximally concave!)

$$\left(\frac{dW}{W} = (r + (\mu - r) u) dt + \sigma u(t) dW_t \right)$$

$$d \log W(t) = \frac{1}{W} dW + \left(-\frac{1}{2W^2}\right)(dW)^2$$

$$= (r + (\mu - r) u) dt + \sigma u \, dW_t \qquad (\text{assume } u : \text{constant})$$
$$- \frac{1}{2} \sigma^2 u^2 dt$$

$$\Rightarrow \log \left(W(t) / W(0) \right) = \Big(\underbrace{\{ r + (\mu - r) u \} - \frac{1}{2} \sigma^2 u^2}_{\text{maximize this}} \Big) t$$
$$+ \sigma u W(t)$$

Max. growth rate $\Rightarrow (\mu - r) . 1 - \sigma^2 u^* \Rightarrow u^* = \frac{\mu - r}{\sigma^2}$

"log utility + constant utility"

$$\Rightarrow \frac{1}{t} \log W(t)/w = r + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 + \frac{\sigma r}{t} W(t)$$

$\rightarrow 0$ as $(t \rightarrow \infty)$

$$\Rightarrow \frac{1}{t} \log W(t)/w \rightarrow r + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2$$

- Power of rebalancing.

Recall $\frac{dX(t)}{X(t)} = \mu \, dt + \sigma \, dW(t)$

Then, $\frac{1}{t} \log \frac{X(t)}{X(0)} \rightarrow \mu - \sigma^2/2$

volatility diminishes drift.

Difficulty in real-world. $\rightarrow$ pairs trading.

Suppose $\mu \sim \sigma^2/2 \rightarrow$ no growth.

optimal allocation, $u^* = \frac{\mu - r}{\sigma^2} = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)$

$r \downarrow \quad \sigma \uparrow$ (volatile) $\rightarrow u^* \sim 1/2$: close to optimal

growth rate : $r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$

$= r + \frac{\sigma^2}{8}\left(1 - \frac{2r}{\sigma^2}\right)^2$

Assume $r/\sigma^2 \ll 1 \Rightarrow \left(r + \sigma^2/8\right)$ is optimal return.

increases with σ

02/12/2026

- μ : modeled as ① random & ② mean-reverting.
  Then, u^* is much smaller. ~ realistic.

- Hamilton-Jacobi-Bellman (HJB) Equation. → stochastic control.

$$dX(t) = b(X(t), u(t))\,dt + \sigma(X(t), u(t))\,dB(t)$$

$(x \in \mathbb{R},\ u \in \mathbb{R})$.

$u(t)$: non-anticipatory : depends only on past up to time t.

$$V(t,x) = \inf_u E_{t,x}\{g(X(T))\} \quad (t<T)$$ → "value function."

↓ (conditional : $X_t = x$.)

$$\mathcal{L}_u = \frac{1}{2}\sigma^2(x,u)\,\partial^2/\partial x^2 + b(x,u)\,\partial/\partial x$$ → "infinitesimal generator."

$V(t,x)$ solves HJB eq.,

$$\partial V/\partial t\,(t,x) + \inf_u\{\mathcal{L}_u V(t,x)\} = 0 \quad (t<T)$$

$$V(T,x) = g(x) \quad (t=T)$$

* Solution

Suppose $V(t,x)$ exists and apply Itô's formula.

→ What is optimal u now?

$\partial V/\partial t + \inf_u \{ \mathcal{L}_u V \} = 0 \quad \to \quad u^*(t,x)$?

Assume u^* is unique and differentiable, $\to$ solves the SDE!

$$dX^*(t) = b(X^*(t), u^*(t, X^*(t))\,dt$$

$$+ \sigma(X^*(t), u(t, X^*(t)))\,dB(t)$$

$\to$ $X^*(t)$ is optimal state

$u^* = u^*(t, X^*(t))$ is optimal control.

Where HJB comes from? $\to$ "Verification lemma."

If $V(t,x)$ is smooth solution of HJB,

then $V(t,x)$ is value function of control problem,

and $u^* = u^*(t, X^*(t))$ is optimal control.

pf) Let $u(t)$ is control and $X^u(t)$.

$$dX^u(t) = b(X^u(t), u(t))\,dt + \sigma(X^u(t), u(t))\,dB(t)$$

Apply Ito: $dV(t, X^u(t)) = \partial V/\partial t\, dt + \partial V/\partial x\, dX^u$

$\searrow = b\,dt + \sigma\,dB(t)$.

$$+ \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dX^u)^2$$

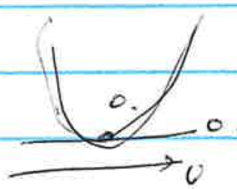
$\downarrow \sigma^2 dt$.

$$= \Big(\partial V/\partial t + \underbrace{\tfrac{1}{2}\sigma^2 \partial^2 V/\partial x^2 + b\,\partial V/\partial x}_{= \mathcal{L}_u V}\Big)dt + \sigma \frac{\partial V}{\partial x}\,dB(t)$$

$\Rightarrow$ Integrate,. $V(T, X^u(T)) = V(t, X^u(t)) + \int_t^T (\partial V/\partial s + \mathcal{L}_u V)\,ds + \int_t^T \sigma \frac{\partial V}{\partial x}\,ds$

The integral leads to,

Recall that, $\partial V/\partial t + (\mathcal{L}_u V)_{\inf} = 0$



$\Rightarrow \partial V/\partial t + \mathcal{L}_u V \big|_{u(t)} \geq 0 \rightarrow$ not infimum. —①

Take expectation,

$$E_{t,x}\{ V(T, X^u(T))\} \geq V(t,x) \qquad (\text{from ①})$$

$$V(t,x) \leq E_{t,x}\{ g(X^u(T))\}.$$

Committor? Bdry Conditions?

$$\Rightarrow V(t,x) \leq \underbrace{\inf_u E_{t,x}\{ g(X^u(T))\}}$$

$=$ Value Function

$X^u(t) \rightarrow X^*(t)$
$u(t) \rightarrow u^*(t, X^*(t))$
} substitute, get equality (exact)

04/18/2026.

- Book Keeping

$$Q_t = \underbrace{Q_t^b}_{\text{buy}} - \underbrace{Q_t^a}_{\text{sell}} \quad \text{(asking to)}$$

X_t = cash on hand. (X_0 : cash to start with)

$$dX_t = P_t^a \cdot \underbrace{dQ_t^a}_{\text{sell}} - P_t^b \underbrace{dQ_t^b}_{\text{buy}}$$

Spreads : $\delta_t^a = \underbrace{S_t}_{\text{observed mid-price (spot)}} - P_t^a$, $\delta_t^b = S_t - P_t^b$

Model : $S_t = S_0 + \sigma W_t$ (scale of minutes, seconds)

→ Is this good model? Depends on 'time scale'

What are Poisson inventory processes?

$Q_t^a \sim$ Poisson with intensity $\lambda^a(t) = A e^{-k\delta_t^a}$

$Q_t^b \sim$ " $\lambda^b(t) = B e^{-k\delta_t^b}$

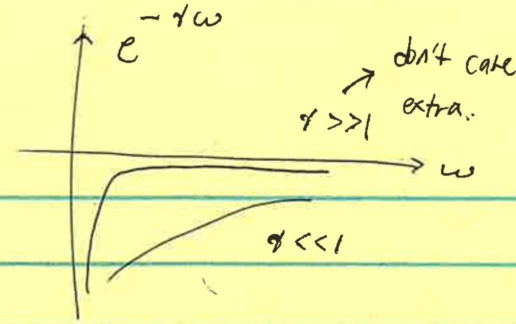
Wealth process : $\mathcal{W}_t = X_t + Q_t S_t$ (Q_t may be < 0, > 0 → holding stock.)

Let $h(s, q, x) = e^{-\gamma(\underbrace{x + qs}_{\text{wealth}})}$

= expected utility of portfolio wealth.

$$T > t, \; E\left[e^{-\gamma \mathcal{W}_T} \;\middle|\; S_t = s, X_t = x, Q_t = q \right]$$

$$u(t,s,x,q) = \sup_{\delta^a,\delta^b} E\{-e^{-\gamma IN(t)}\}$$



val. func. opt. control, δ^a δ^b are control variables.

Short Term Trading. → only interested in fluctuations. 02/19/2026.

S_t = Visible mid-price (at an index)
$= S_0 + \sigma W_t$ (just B.M.) "Goal : Buy low sell high"

Q_t^a = # of sell (ask) trades up to time t
Q_t^b = # of buy (bid) trades up to time t
Assume, # of trades restrict to 1.

$\boxed{\text{Inventory}} \Rightarrow Q_t = Q_t^b - Q_t^a + q$ ↗ start! → trading @ t.

X_t : cash on hand.

$\boxed{\text{Book Keeping}}$: $dX_t = p_t^a \, dQ_t^a - p_t^b \, dQ_t^b$
↳ sale price. ↳ buy price.

$\boxed{\text{Spread}}$: $s_t^b = S_t - p_t^b \quad (\geq 0)$
$s_t^a = p_t^a - S_t \quad (\geq 0)$

Assumption : $Q_t^a \sim \text{Poisson}\left(\lambda = \lambda^a(t) = A\, e^{-k\delta_t^a}\right)$ ✓
$Q_t^b \sim \text{Poisson}\left(\lambda = \lambda^b(t) = A\, e^{-k\delta_t^b}\right)$ ✓

$\{W_t, Q_t^a, Q_t^b\}$ are statistically independent.

W (wealth) $= X_t + Q_t S_t$, $U(w) = -e^{-\gamma w}$ (utility)
→ for convenience.

- Control Problem

→ Determine δ_t^a, δ_t^b $(0 \le t \le T)$

$$\sup_{\delta_t^a, \delta_t^b} \mathbb{E}\{U(W_T) | \mathcal{F}_0\} \rightarrow \text{optimal } \delta_t^{a*}, \delta_t^{b*}$$

Introduce value function,

$$\rightarrow u(t, s, q, x) = \sup_{\delta a, \delta b} \mathbb{E}\{U(W_T) | S_t = s, Q_t = q, X_t = x\}$$

show that u satisfies HJB, (PD-DE equation)..

Risk aversion → $-e^{-\gamma w}$ (if I have \$1, \$1 more is huge; " \$100, \$1 " is small)

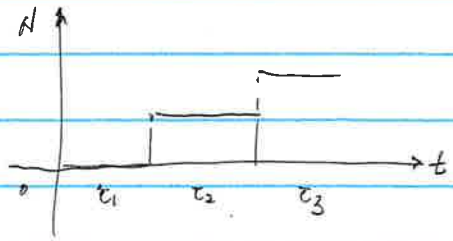
↳ $\gamma \propto$ how much difference?

Liquidity → $\lambda \propto e^{-k(S_t - P_t^a)}$ (k is liquidity measure) ($k \downarrow$, don't change a lot)

(Read : poisson p.p.):

02/25/2026.

- Poisson Process



$\mathbb{E}\{N(t)\} = \lambda t$

λ : intensity / rate.

Define $M(t) = N(t) - \lambda t$: $\mathbb{E}\{M(t)\} = 0$., $\mathcal{F}_t = \sigma\{N(s),\ s \le t\}$

Claim: $\mathbb{E}\{M(t) \mid \mathcal{F}_s\} = M(s)$ (Martingale)

- Ito's formula.

$f: \mathbb{Z}^+ \to \mathbb{R}$, $f(N(t)) - f(N(s)) = \sum_{s < \tau_k < t} \Delta f(N(\tau_k^-))$

$\Delta f(N(\tau_k^-)) = f(N(\tau_k)) - f(N(\tau_k^-))$
$= f(N(\tau_k^-)+1) - f(N(\tau_k^-))$ (τ_k, count increases)

$$f(N(t)) - f(N(s)) = \int_s^t \Delta f(\underbrace{N(\gamma-)}_{\text{jump happens!}})\, dN(\gamma)$$

$$\Rightarrow \quad f(N(t)) - f(N(s)) = \int_s^t \Delta f(N(\gamma-))\lambda\, d\gamma + \int_s^t \Delta f(N(\gamma))\,\lambda\, d\gamma .$$

$$= \int_s^t \Delta f(N(\gamma))\,\lambda\, d\gamma + \underbrace{\int_s^t \Delta f(N(\gamma-))\, dM(\gamma)}$$

$= ?$

$$f(t, N(t)) = f(s, N(s)) + \int_s^t \left(\frac{\partial f}{\partial \gamma} + \lambda \Delta f\right)(\gamma, N(\gamma))\, d\gamma$$
$$+ \int_s^t \Delta f(\gamma, N(\gamma-)).\ dN(\gamma).$$

(Ito!)

- Suppose $u(t,n)$ satisfies

$$\partial u/\partial t + \lambda \Delta u = 0 \quad , \; t < T$$
$$u(T,n) = g(n), \; t = T$$

Then, $u(T, N(T)) = u(t, N(t)) + \int_t^T \Delta u(\gamma, N(\gamma-)) dM$

$\quad \| $

$g(n(T))$ $\quad$ (N is Markovian) ?

$\Rightarrow \mathbb{E}\{ g(N(T)) \mid \mathcal{F}_t \} = u(t, N(t)) + 0$ ($\because$ Martingale)

$\therefore \; u(t,n) = \mathbb{E}_{t,n} \{ g(N(T)) \mid N(t) = n \}$ $\rightarrow$ Markov! Yes!

$\Rightarrow$ satisfies the difference / differential equation. (BKE)

$$\left. \begin{aligned} &\rightarrow \partial u/\partial t + \lambda(t) \cdot \big(u(t,n+1) - u(t,n) \big) = 0 && t < T \\ &\qquad\qquad u(T,n) = g(n) && t = T \end{aligned} \right\}$$

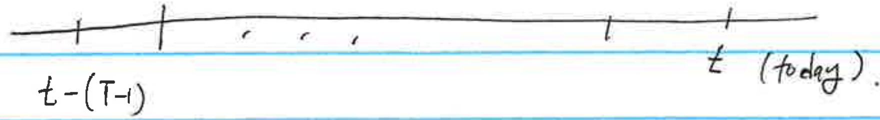
Notes $\rightarrow$ From Ito we can show $N(t) - N(s)$ is Poisson.

- Both $W(t)$ and $N(t)$

$f(t, W(t), N(t)) - f(s, W(s), N(s))$

$$= \int_s^t \left[\frac{\partial f}{\partial \gamma} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \lambda \Delta f(\cdot) \right] d\gamma$$

$$+ \int_s^t \frac{\partial f}{\partial x}(\gamma, W(\gamma), N(\gamma)) \, dW(\gamma) + \int_s^t \Delta f\big(\gamma, W(\gamma), \underbrace{N(\gamma-)}_{\text{"}\gamma-\text{"}}\big) \, dN(\gamma)$$



Return matrix $R = [R_{it}]$ $N \times T$ entries

N: # of stocks; T: # of windows

R = signal + noise $\rightarrow$ P.C.A. $\in \mathbb{R}^{N \times T}$

$$\underset{N\times N}{R = V}\ \underset{N\times T}{\Lambda}\ \underset{T\times T}{U^T} \qquad RR^T \ ?$$

Normalization : ① $R_{it} \leftarrow R_{it} - 1/T \cdot \sum_t R_{it}$

$$\Rightarrow \sum_t R_{it} = 0$$

② $R_{it} \leftarrow \dfrac{R_{it} - 1/T \sum_t R_{it}}{\left[\sum_t \left(R_{it} - 1/T \sum_t R_{it}\right)^2\right]^{1/2}}$ Empirical variance.

Then, $1/T \, RR^T = V \Lambda^2 V^T$ $\rightarrow$ empirical covariance. (SPD)

Order eig.vals of $\Lambda^2 \rightarrow \lambda_1^2 \geq \lambda_2^2, \ldots, \geq \lambda_n^2 > 0$

$$\sum_i \lambda_i^2 = Tr\left(RR^T/T\right) = \sum_{i=1}^{N} \left(\frac{R_{it} - \bar{R}_i}{\sigma_i}\right)^2 = 1/T$$

$$= N$$

The eigenvalues have $\sum_{i=1}^{N} \lambda_i^2 = N$

λ_i^2 is measure of variance of i^{th} to the total variance.

Suppose 'p' significant eigenvalues.

'p' is a measure of diversity.

$R = \underset{\sim}{L}\underset{\sim}{F} + \rho \longrightarrow$ What is ρ ? $\rightarrow$ Is this MP distribution?

$V\Lambda U^T$ factor!

$\boxed{No!}$

$\because$ p $\uparrow$ $\rightarrow$ okay distribution.

Marchako - Paston Thm (1967)

$X \in \mathbb{R}^{m \times n}$, iid entries with σ^2 variance $\rightarrow$ random matrix

$Y_m = 1/m \cdot X X^T$

Let $\lambda_1 \sim \lambda_m$ are eig-vals of Y_m, $\mu_m(A) = 1/n \, \#(\lambda_i \in A)$

↳ empirical dist. eig vals.

As $n \to \infty$, $m \to \infty$, $m/n \to \lambda$, $0 \le \lambda < \infty$,

Then, $\mu_m(A) \to \mu(A)$.

$$\mu(x) = \begin{cases} (1 - 1/\lambda)\, \delta_0(x) + \nu(x) & , \lambda > 1 \\ \nu(x) & , 0 \le \lambda \le 1 \end{cases}$$

$$\nu(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{\lambda^+} \quad (\lambda_- \le x \le \lambda_+), \quad \lambda_\pm = \sigma^2(1 \pm \sqrt{\lambda})^2$$

$$E[\nu] = \sigma^2, \quad Var(\nu) = \sigma^4 \lambda$$

Ex) m = const. $n \to \infty \Rightarrow \mu(x) = \sigma^2 \pm 0$