- Introduction. 01/09/2024

→ Systems that evolve in space & time are described by PDEs.

{ PDE : An equation that relates a multivariate function ψ and its partial derivatives in 2 or more independent variables. }

- ψ is tensor $\begin{cases} \text{Scalar} \\ \text{vector} \\ \text{tensor} \end{cases}$ example (Advection) $\begin{cases} \partial \rho/\partial t + u \cdot \partial \rho/\partial x = 0 & (1D) \\ \partial \rho/\partial t + (\vec{u} \cdot \nabla)\, \rho = 0. & (ND) \end{cases}$

- check linearity of a solution : if $\mathcal{L}(\phi)$ is solution → check if $\mathcal{L}(\alpha\phi_1 + \beta\phi_2) \equiv \alpha\mathcal{L}(\phi_1) + \beta\mathcal{L}(\phi_2)$.

- conservation (ND) → (1) $\underline{\partial\phi/\partial t + \nabla \cdot \vec{F} = 0}$ $\left(\vec{F} = \vec{F}(\psi)\right)$

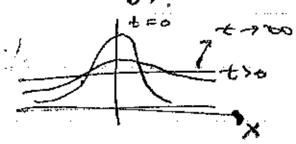
- Non-linear Advection eqn. (Burger's eqn.) → (2) $\underline{\partial u/\partial t + u\, \partial u/\partial x = 0}$ (inviscid Burgers)

  ↳ Use "Characteristics methods". ↓ Non-linear P.D.E.

- Diffusion eqn. → (3) $\underline{\partial T/\partial t = \alpha\, \partial^2 T/\partial x^2}$ $\left(\alpha = \frac{k}{\rho C_p} = \text{thermal diffusivity}\right)$

  ↓ Heat, Mass, Concentration, Momentum, ...

  ↳ $\partial T/\partial t = \alpha \nabla^2 T$ (ND)



- Derivation of P.D.E.

$(dx\,dy)\,\rho C_p\, \partial T/\partial t = (F_x - F_{x+dx})\, dy$ (flux) (going in is +).
$\qquad + (F_y - F_{y+dy})\, dx$

(Fourier's law : $F_x = -k\frac{\partial T}{\partial x}$)

$\Rightarrow -\rho C_p\, \partial T/\partial t = \{\partial F/\partial x + \partial F/\partial y\}$

$\Rightarrow -\rho C_p\, \partial T/\partial t = -k\left(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2\right) = -k \nabla^2 T.$

$\Rightarrow \underline{\partial T/\partial t = \alpha \nabla^2 T}$ (3) ✓

We can also argue. $\rho C_p\, \partial T/\partial t + \nabla \cdot \vec{F} = 0$ ($\boxed{\text{Conservation law}}$). (1)

$\Rightarrow \rho C_p\, \partial T/\partial t + \nabla \cdot (-k\nabla T)$

$\Rightarrow \underline{\partial T/\partial t = \alpha \nabla^2 T}$ (3) ✓

$\partial T/\partial t = \alpha \nabla^2 T \xrightarrow{\text{Steady state}} \nabla^2 T = 0$ (Laplace eqn.)

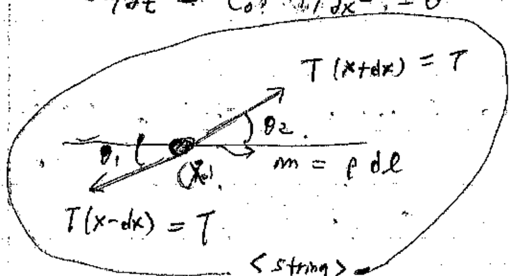
$\nabla^2 T = S$ (Poisson's eqn.)

S ↓ Heat source

- Vibration / waves.

$\partial^2 u/\partial t^2 - C_0^2\, \partial^2 u/\partial x^2 = 0 \rightarrow \partial^2 u/\partial t^2 - C_0^2 \cdot \nabla^2 u = 0$ (4) (d'Alembert's eqn).

(wave equation)



- Classification of PDE. $\phi(x,y)$ for examples

① order / degree

First order : $a(\phi, x, y)\, \partial\phi/\partial x + b(\phi, x, y)\, \partial\phi/\partial y + c(\phi, x, y) + d(x,y) = 0$

Second order : $A(\partial\phi/\partial x, \partial\phi/\partial y, \phi, x, y)\, \phi_{xx} + B(\cdots)\, \phi_{xy} + C(\cdots)\, \phi_{yy} = 0$.

(First order terms $= 0$)

How to discriminate ?

Suppose $A^2+B^2+C^2 \neq 0$, ① $B^2-4AC > 0$: Hyperbolic (wave eqn)

⇒ information traveling

② $B^2-4AC = 0$: Parabolic (diffusion)

③ $B^2-4AC < 0$: Elliptic (Laplace, poisson)

(1) Linear (2) Homogeneous (3) Init. Cond. / Bound. Cond. → well-posedness.

↓

Depends on domain / indep. variables.

• Solution methods:	Linear	Non-linear
1) Characteristics	o	o
2) Seperation of var.	o	×
3) Integral transformation	o	×
4) similarity solution,	o	o

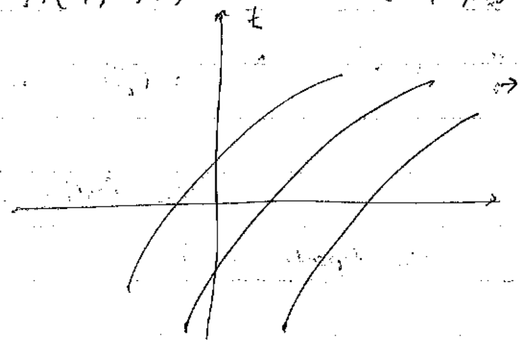
superposition { 2), 3)

Goal : PDE → ODE

First order / ... order PDEs 01/11/2024

- First order PDE.

$$A(\phi, x, t)\,\partial\phi/\partial t + B(\phi, x, t)\cdot\partial\phi/\partial x + C(\phi, x, t) + D(x, t) = 0 \quad \text{——} ①$$



→ family of characteristics.
(alongside PDE → ODE)

$\phi(x, t)$ on trajectory of $\phi(x(s), t(s))$

$$\therefore\ \partial\phi/\partial s = \partial\phi/\partial x\ \partial x/\partial s + \partial\phi/\partial t\cdot\partial t/\partial s \quad \text{——} ②$$

if $A \neq 0$ ⇒ ② → ① ⇒ $\partial\phi/\partial t + B/A\ \partial\phi/\partial x + C/A + D/A = 0$ —— ③

$$\Rightarrow 0 = \partial\phi/\partial x\,\{\partial x/\partial s - B/A\cdot\partial t/\partial s\} - \frac{C+D}{A}\cdot\partial t/\partial s = \partial\phi/\partial s$$

$$\Rightarrow \cancel{\partial\phi}\ \frac{d\phi}{dx}\left\{\frac{dx}{ds} - \frac{B}{A}\frac{dt}{ds}\right\} - \frac{C+D}{A}\frac{dt}{ds} = \frac{d\phi}{ds} \quad \text{——} ④$$

③ also becomes d/dt form

Choose $(s = t)$ ⇒ $dt/ds = 1$, $dx/dt = v$.

For ④ to hold, $\frac{dx}{ds} - \frac{B}{A}\frac{dt}{ds} = 0$ and $\frac{d\phi}{ds} = -\frac{C+D}{A}\frac{dt}{ds}$

∵ It has to satisfy for arbitrary (s)

$$\therefore\ \frac{dx}{dt} = \frac{B(\phi, x, t)}{A(\phi, x, t)} \quad \text{and} \quad \frac{d\phi}{dt} = -\frac{C(\phi, x, t) + D(x, t)}{A(\phi, x, t)}$$

PDE → 2 distinct ODEs : These are called,
Family of characteristic curves, $(x(s), t(s), \phi(s))$

As information (s) flows. (information flow occurs),
the ϕ function evolves through the (information) flow.

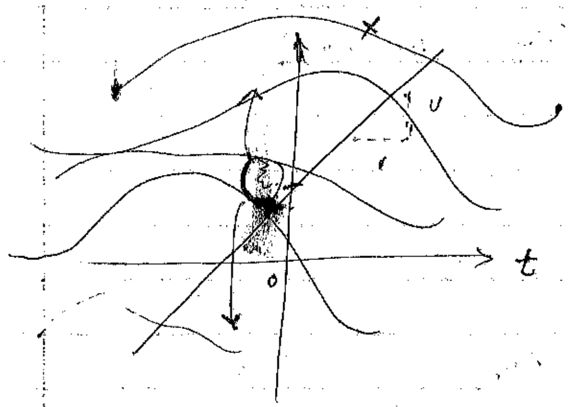
$\xi \quad \xi$

Example.

① $\partial \psi/\partial t + U \cdot \partial \psi/\partial x = 0$ $\Rightarrow$ Apply $\Rightarrow$ $\frac{dx}{dt} = U/1 \Rightarrow x = Ut + \xi$

$\begin{pmatrix} -\infty < x < \infty \\ 0 < t < \infty \end{pmatrix}$ $\therefore \frac{d\psi}{dt} = 0 \Rightarrow \psi = F$ constants.

same for (x, t)



$\therefore$ F only depends on ξ.

$F(\xi(x_0, t=0)) = F(x - Ut) = e^{-x^2}$

Initial condition

② $\partial u/\partial t - x/2 \cdot \partial u/\partial x = 0$ and $u(x, t=0) = e^{-x^2}$

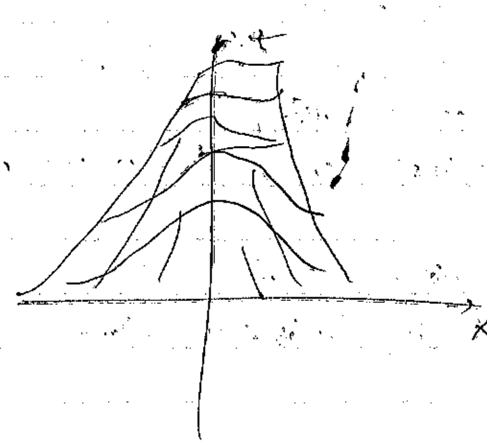
On characteristics, $\left.\frac{dx}{dt}\right|_\xi = -\frac{x}{2}$ (1) and $\left.\frac{du}{dt}\right|_\xi = 0$ (2)

(1) $\Rightarrow$ $\frac{1}{x}dx = -\frac{1}{2}t \Rightarrow \ln|x| = -\frac{1}{2}t + c$

$\Rightarrow x = e^c \cdot e^{-1/2 \cdot t} = \xi e^{-t/2}$.

(2) $\Rightarrow$ $u = F(\xi)$ (constant) $= F(x \cdot e^{t/2})$

$u(x, t=0) = F(x) = e^{-x^2} \Rightarrow u = e^{-\xi^2}$



Example 3) $\partial u/\partial t + t\,\partial u/\partial x = 0 \quad u(x, t=0) = e^{-x^2}$

Example 4) $\partial u/\partial t + \partial u/\partial x = u \quad u(x, t=0) = e^{-x^2}$ (linear) (Homogeneous)

$\Rightarrow$ Along characteristics, $\left.\frac{dx}{dt}\right|_{\xi} = 1$ and $\left.\frac{du}{dt}\right|_{\xi} = -u$

$\Rightarrow \quad x = t + c = \underline{t + \xi}$ ($\because$ At time $=0$ we want ξ)

$\Rightarrow \quad u = Ae^{-t} = u_0\, e^{-t} = \underline{u_0(\xi)\, e^{-t}}$

$$u(x, t=0) = u_0(\xi) = e^{-x^2} = e^{-\xi^2}$$

$$\Rightarrow \quad u = e^{-\xi^2} e^{-t} \quad \text{and} \quad x = t + \xi.$$

$$\Rightarrow \quad \underline{u(x,t) = e^{-(x-t)^2} e^{-t}}$$

Example 5) — Burger's equation.

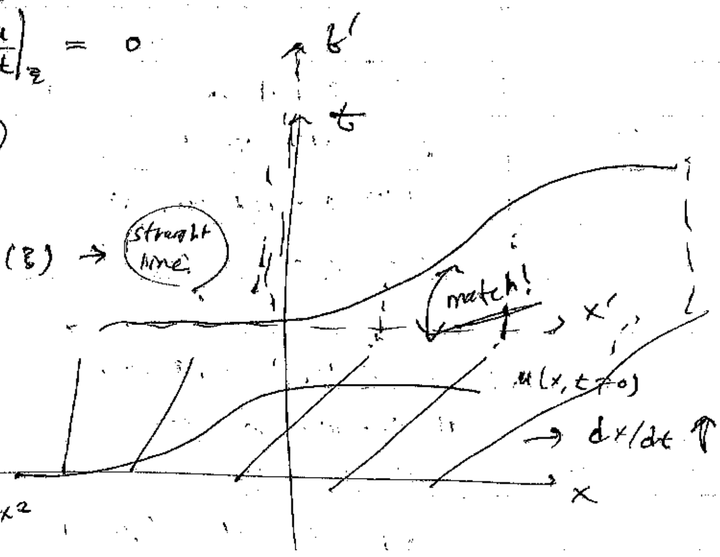
$\partial u/\partial t + u\,\partial u/\partial x = 0 \qquad u(x, t=0) = e^{-x^2}$ $u(x(t), t)$

Along characteristics, $\left.\frac{dx}{dt}\right|_{\xi} = u \qquad \left.\frac{du}{dt}\right|_{\xi} = 0$

$\Rightarrow \underline{u = F(\xi)}$ → u is independent of time

slope of characteristics doesn't change. $u(\xi) \rightarrow$ straight line

$x = ut + \xi = F(\xi)\, t + \xi$



$\underline{u = F(x - ut)}\Big|_{x, t=0} = F(x) = e^{-x^2}$

$\Rightarrow$ $u = e^{-(x-ut)^2}$ → "Implicit" solution

$k_b = 2.63\,e\,5$
$k_s = 3.37\,e{-1}$

Some Notes.

① Linearity – only talks about y respecto to x

Eg. $y'' + xy + x^3 y = x^5 \rightarrow$ Linear. (y 선형).

② Homogeneity $\rightarrow Q y'' + Q y = 0$. if it can be written as

$\underline{\sum a_\alpha(x) D^\alpha y = f(x) = 0}$ $\rightarrow$ Homogeneous.

* Exact equations.

$M(x,y) + N(x,y)\, dy/dx = 0$

$\psi(x,y) \rightarrow \partial\psi/\partial x = M = \psi_x \Rightarrow \underbrace{\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}}_{= \frac{d\psi}{dx}} = 0 \rightarrow \psi(x,y) = 0$

$\partial\psi/\partial y = N = \psi_y$

$\Rightarrow M_y = N_x \Rightarrow \underline{\psi = \int M dx \text{ or } \psi = \int N dy}$

$\psi_{xy} = \psi_{yx}$

③ 2^{nd} order $ay'' + by' + cy = g(t)$

Homogeneous

1) Constant coeff & linear $\rightarrow y = e^{rt} \Rightarrow \underbrace{(ar^2 + br + c)}_{\text{characteristic equation}} e^{rt} = 0$

(i) 2 real roots

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

(ii) two complex conj roots ($r = \alpha \pm j\beta$)

$y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

(iii) Repeated root r

$y = c_1 e^{rt} + c_2 t e^{rt}$

Inhomogeneous

$y = y_h + y_p \rightarrow$ particular.

① Undetermined coeffs.

$y'' - y = 3t^2 + t + 1$

$y_p = at^2 + bt + c$ (guess)

② Variation of parameters.

$y_p = u_1 y_1 + u_2 y_2$

$u_1 = -\int \frac{y_2 g(t)}{W(y_1, y_2)} dt$

$u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$

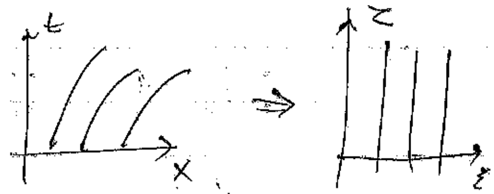
$W = y_1 y_2' - y_2' y_1'$

01/16/2024

PDE — characteristics, curl, shocks,

$$A(\psi,x,t)\,\partial\psi/\partial t + B(\psi,x,t)\,\partial\psi/\partial x + \tilde{C}(\psi,x,t) = 0 \quad (*)$$

$$\tilde{C}(\psi,x,t) = C(\psi,x,t) + D(x,t)$$



$$(x,t) \to (\xi(x,t),\ \tau(x,t))$$

$$\partial\psi/\partial t = \partial\psi/\partial\tau\ \partial\tau/\partial t + \partial\psi/\partial\xi\ \partial\xi/\partial t$$
$$\partial\psi/\partial x = \partial\psi/\partial\tau \cdot \partial\tau/\partial x + \partial\psi/\partial\xi \cdot \partial\xi/\partial x$$

$\Rightarrow$ (*) becomes $\left(A\,\partial\tau/\partial t + B\,\partial\tau/\partial x\right)\partial\psi/\partial\tau + \left(A\,\partial\xi/\partial t + B\,\partial\xi/\partial x\right)\partial\psi/\partial\xi = \ldots$

Need PDE $\to$ ODE along $\underline{\xi(x,t)}$.

Good choice!

• $\tau = t$ $\to$ This will work for most cases.

$$\Rightarrow A\,\partial\psi/\partial\tau + \underbrace{\left(A\,\partial\xi/\partial t + B\,\partial\xi/\partial x\right)}_{=0}\partial\psi/\partial\xi = -\tilde{C}$$

we want $\partial\xi = 0 = \partial\xi/\partial x\,dx + \partial\xi/\partial t\,dt = 0 \Rightarrow \partial\xi/\partial t = -\left.\dfrac{dx}{dt}\right|_{\xi}\dfrac{\partial\xi}{\partial x}$

Along characteristic, $A\left(-\left.\dfrac{dx}{dt}\right|_{\xi}\dfrac{\partial\xi}{\partial x}\right) + B\left(\dfrac{\partial\xi}{\partial x}\right) = \dfrac{\partial\xi}{\partial x}\left(-A\left.\dfrac{dx}{dt}\right|_{\xi} + B\right)$

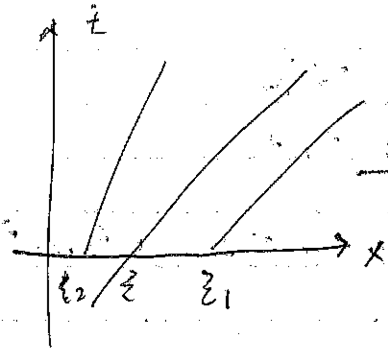
$$\Rightarrow \underline{\left.\frac{dx}{dt}\right|_{\xi} = \frac{B}{A}} \quad \text{and} \quad \underline{\left.\frac{\partial\psi}{\partial\tau}\right|_{\xi} = \frac{-\tilde{C}}{A}}$$

Burger's equation

1st order, Non linear PDE ; $\partial x/\partial t + u \frac{\partial u}{\partial x} = 0$

$$\left.\frac{dx}{dt}\right|_\xi = u , \quad \left.\frac{du}{dt}\right|_\xi = 0 \Rightarrow u(x,t) = G(\xi) \Rightarrow \text{characteristics are straight lines}$$

$$\left. dx/dt \right|_\xi = \text{const}$$



$$\underline{x = G(\xi)\,t + \xi}$$

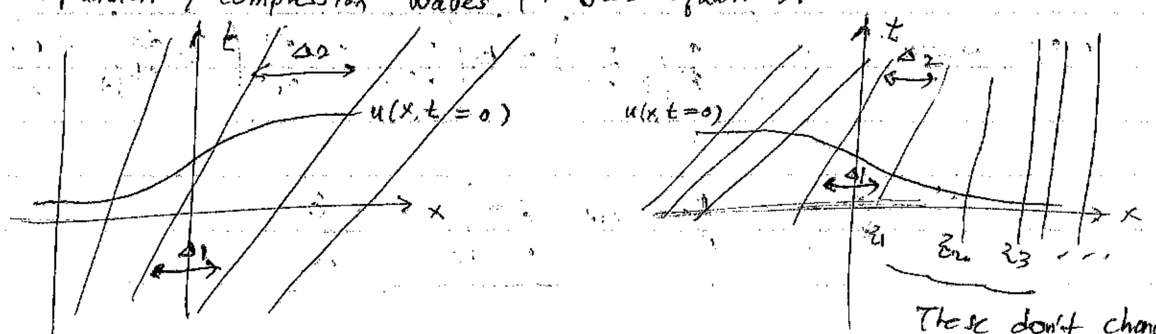
$$\xi = x - G(\xi)\,t$$

Impose initial condition : $u(x, t=0) = G(x - \cancelto{0}{G(\xi)t}) = G(x) = F(x)$

Solution : $\underline{u(x,t)} = F(\xi) = \underline{F(x - ut)}$

$\equiv$ implicit equation for u.

- Expansion / Compression waves (Burger's equation).

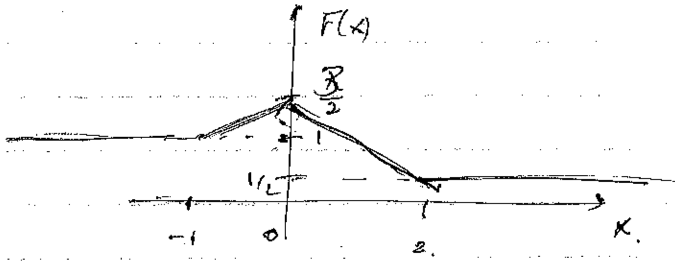


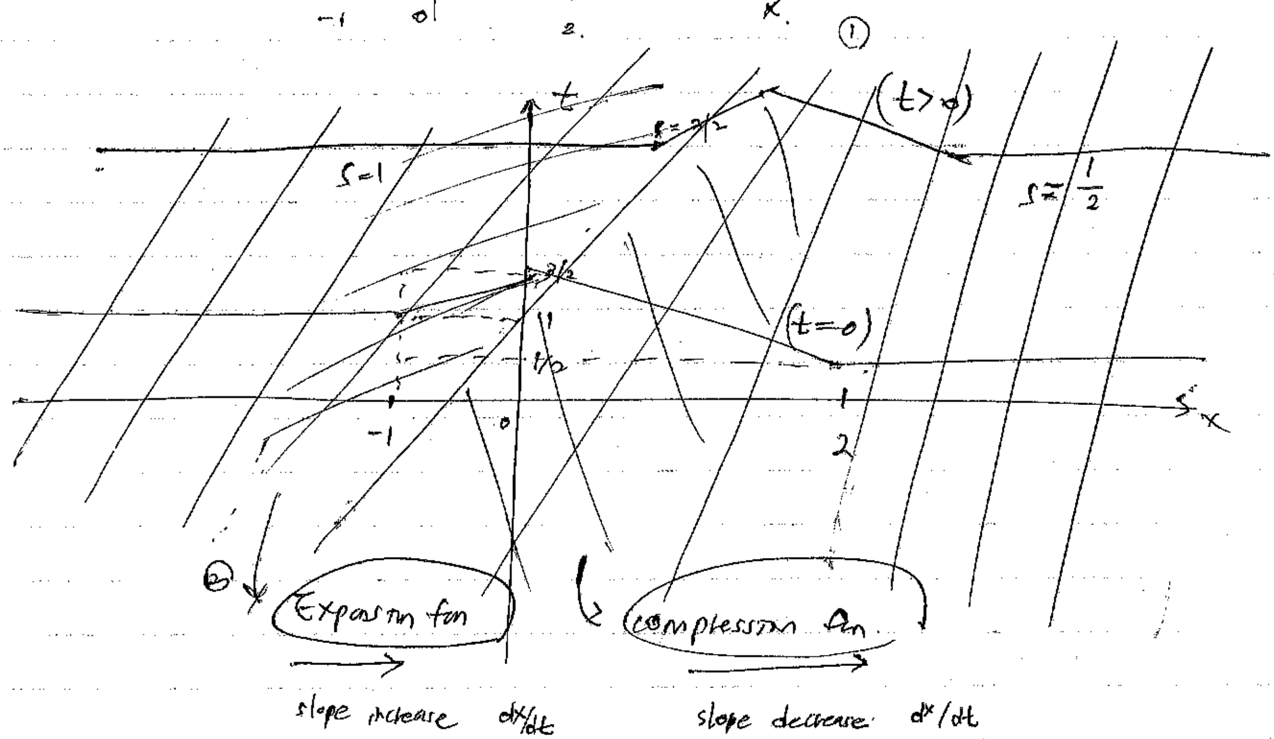
$\Delta 1 < \Delta 2$ (expansion) $\quad\quad$ $\Delta_1 > \Delta_2$ (compression).

Example

$\partial u/\partial t + u\, \partial u/\partial x = 0 \qquad u(x, t=0) = F(x) = \begin{cases} 1 & x<-1 \\ \frac{x}{2}+\frac{3}{2} & -1\le x<0 \\ \frac{3}{2}-\frac{x}{2} & 0\le x<2 \\ \frac{1}{2} & 2\le x<\infty \end{cases}$

u



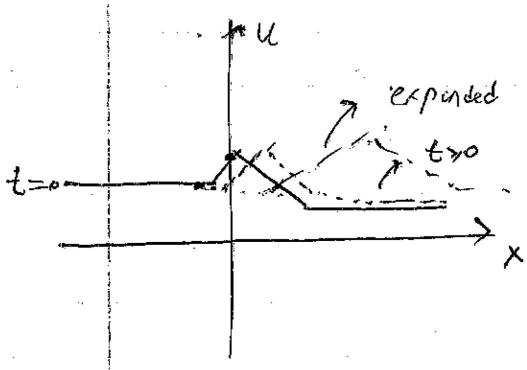


① converges to a point. ⟶ t>0 (shock happens)

② converges to a point. ⟶ t<0

↓

t = shock formation time.



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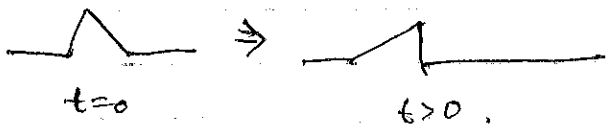
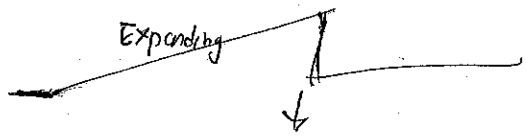


⇓



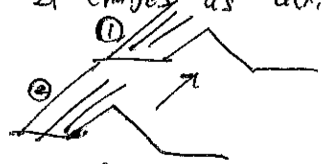
discontinuity =

It is important that ξ is invariant respect to t

Characteristic equations are drawn for $t = 0$

↓

It changes as $u(x,t)$ evolves



① & ② are different!

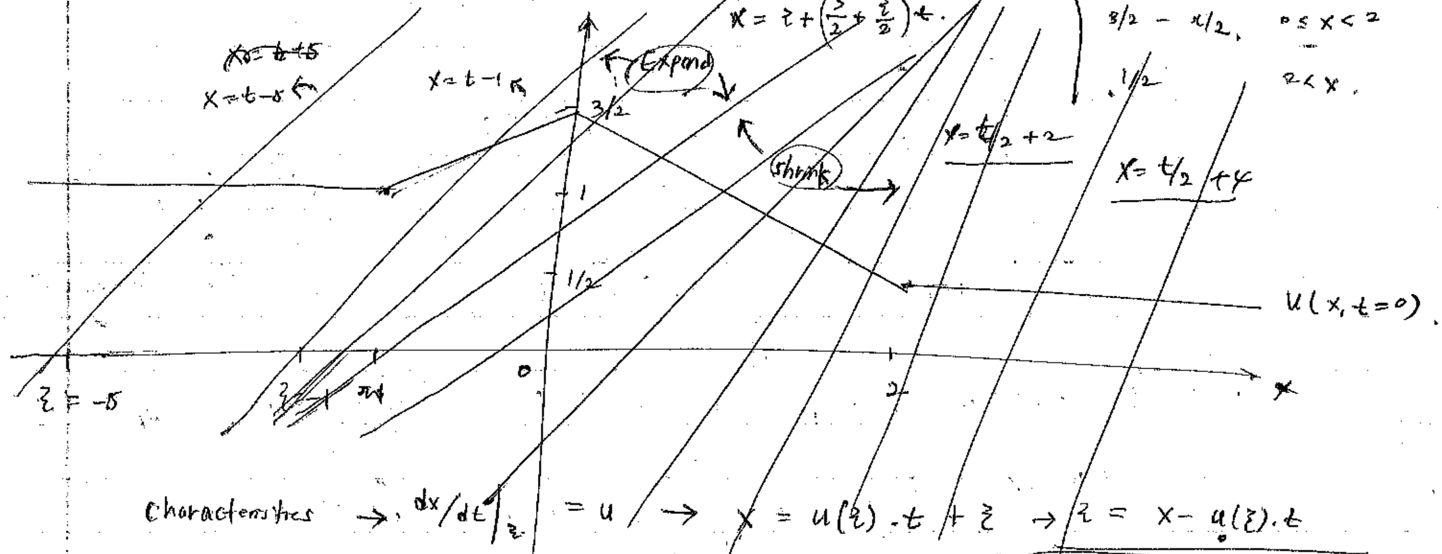
01/18/2024

pb #3 → start from $\vec{n}=1$.

- Lecture 4 : shocks.

→ Burger's equation $\partial u/\partial t + u\, \partial u/\partial x = 0$

$$\overbrace{u(x,t=0)}^{u_0(x)} = \begin{cases} 1 & x<-1 \\ 3/2 + x/2 & -1\le x<0 \\ 3/2 - x/2, & 0\le x<2 \\ 1/2 & 2<x. \end{cases}$$



characteristics → $dx/dt\big|_{\xi} = u$ → $x = u(\xi)\cdot t + \xi$ → $\xi = x - u(\xi)\cdot t$

$du/dt\big|_{\xi} = 0 \Rightarrow u = F(\xi) = u_0(\xi) = u_0(x - u(\xi)t)$

$u(x, t=0) = F(\xi(x,t=0)) = F(x) = u_0(x)$

$\boxed{\text{Note that this is } u_0(\xi) \text{ not } u(\xi)}$ $\Rightarrow F(\xi) = u_0(\xi)$

(i) $\xi < -1$ $\quad u_0(\xi) = 1 \Rightarrow x = t + \xi.$

(ii) $2 < \xi$ $\quad u_0(\xi) = \frac{1}{2} \Rightarrow x = t/2 + \xi.$

(iii) $-1 \le \xi < 0$ $\quad u_0(\xi) = 3/2 + \xi/2 \Rightarrow x = \xi + u_0(\xi)t = \xi + \left(\frac{3}{2} + \frac{\xi}{2}\right)t$.

(iv) $0 \le x < 2$ $\quad u_0(\xi) = 3/2 - \xi/2 \Rightarrow x = \xi + u_0(\xi)t = \xi + \left(\frac{3}{2} - \frac{\xi}{2}\right)t$

↳ (iii) Expansion zone $\quad -1 \le \xi < 0 \Rightarrow t-1 \le x < \frac{3}{2}t$

$$u(x,t) = u_0(\xi) = \frac{3}{2} + \frac{x/2}{1+t/2} - \frac{3\ t/2}{1+t/2}.$$

(iv) shrink zone. $0 \le \xi < 2 \Rightarrow$ ~~$3/2\,t \le x \le t/2$~~

$\xi = \frac{x - 3/2\,t}{1 - t/2}$ $\quad$ Q) what if $t = 2$?

At $t = 2$ → $x = \xi(1 - t/2) + \frac{3}{2}t = \frac{3}{2}t$

$u =$

→ continued

At singular point, $u(x,t) = u_0(\xi) = (3/2 - \xi/2)\,t + \xi$

$$= 3/2 - \frac{x - 3/2\,t}{2-t} = \frac{3}{2} - \frac{x}{2-t} + \frac{\frac{3}{2}t}{2-t}$$

$x = 3,\ t = 2 \to u(x,t)$ has multiple values..?

$$u(x,t) = \begin{cases} 1 & x < -1+t \\ u_1 & t-1 \le x < \frac{3}{2}t \\ u_2 & \frac{3}{2}t \le x < 2 + t/2 \\ 1/2 & x \ge 2 + t/2 \end{cases}$$

$t < 2$ → shock formation time

- Shock formation time. $du/dx\ (x=3, t=2) = \infty$ idea → applicable anything!

↳ where $\partial u/\partial x$ blows up. $u(x,t) = \overset{u_0}{F}(x - u(x,t)\,t)$

$$\Rightarrow \partial u/\partial x = \partial F/\partial \xi \ \partial \xi/\partial x = \partial F/\partial \xi \cdot (1 - \partial u/\partial x \cdot t)$$

$$\Rightarrow \left(1 + \partial F/\partial \xi\, t\right)\frac{\partial u}{\partial x} = \partial F/\partial \xi \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial F/\partial \xi}{1 + t \cdot \partial F/\partial \xi}$$

$$\Rightarrow t_{shock} = \frac{1}{-\partial F/\partial \xi} \quad \text{(assume } F \text{ is smooth function)}$$

$$t_s = \min_{\xi}\left(-\frac{1}{\partial F/\partial \xi}\right)$$

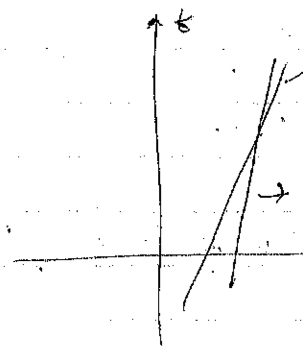
* Formal way

$x = u_0(\xi)\,t + \xi$

$\left.\frac{dx}{d\xi}\right|_{t=t_s} = 0$ (x doesn't change w.r.t. ξ)

$\xi, \xi + d\xi, 2\xi, \dots$ → don't change.

- Shock formation time (geometric interpretation).



$x = \xi + u(\xi)t$

$x = \xi + d\xi + u(\xi + d\xi)\,t$

→ Intersection

$$\Rightarrow d\xi + t_s\left(u(\xi + d\xi) - u(\xi)\right) = 0$$

$$\Rightarrow t_s = \frac{-1}{\frac{u(\xi + d\xi) - u(\xi)}{d\xi}}$$

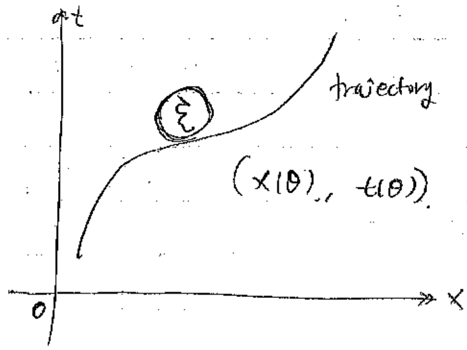
$u_0 \to$ smooth $\to t_{shock} = \frac{-1}{\partial u_0/\partial \xi}$

TA session 01/19/2024

- 1st order PDEs.

$$\partial\phi/\partial t + u(\phi, x, t)\, \partial\phi/\partial x = S(\phi, x, t) \quad ① \qquad x \in \mathbb{R} \quad t \in \mathbb{R} \quad t > 0.$$

I.C. $\phi(x, t=0) = F(x)$.



① $dx/dt\big|_{\text{trajectory}} = u(\phi, x, t)$

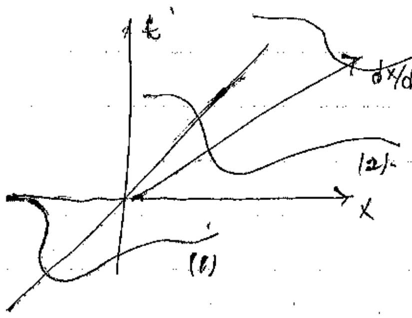
② $d\phi/dt\big|_{\text{trajectory}} = S(\phi, x, t)$

$\because\ d(\cdot)/d\theta = \frac{dt}{d\theta}\, \partial/\partial t + \frac{dx}{d\theta}\, \partial/\partial x \rightarrow$ Apply to ①

↳ $\boxed{\theta = t} \Rightarrow \frac{d}{dt}\Big|_{\text{trajectory}} = 1 \cdot \partial/\partial t + \frac{dx}{dt}\Big|_{\text{trajectory}} \cdot \partial/\partial x$.

- Along the characteristics.

$\frac{dx}{dt}\Big|_{\xi} = u(\phi, x, t) \rightarrow$ Propagation speed of equation. (carry information).



$dx/dt > 0$ (direction of information propagation).

Be careful for dx/dt magnitude & direction.

- $dx/dt = u = \text{constant} \rightarrow$ straight lines

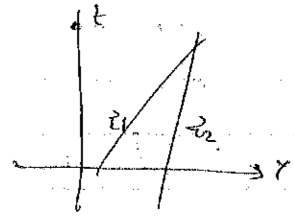
$dx/dt = u(\phi) \rightarrow$ not parallel but straight lines.

Ex.1) $2xt\, \partial\phi/\partial x + (1+t^2)\cdot(\partial\phi/\partial t + \phi) = 0$

Type.
- → First order in x and t.
- → Homogeneous ($\because\ \phi = 0$ satisfies given PDE)
- → Linear equation. ($\because$ No $\phi \cdot d\phi$ terms) → No shock!
- → Variable coefficient.

$\phi(x, t=0) = \tanh(x)$

$$\Rightarrow \partial\phi/\partial t + \underbrace{\frac{2t}{1+t^2}x}_{\boxed{u(x,t)}}\,\partial\phi/\partial x = -\phi$$



$dx/dt\big|_{\xi_1} \neq dx/dt\big|_{\xi_2}$ → shock happens

But, $dx/dt\big|_{\xi_1} = dx/dt\big|_{\xi_2} = u(x,t)$ → uniquely defined.

$dx/dt = \frac{2t}{1+t^2}x \Rightarrow x = \xi(1+t^2)$

$d\phi/dt = -\phi \Rightarrow \phi = \underbrace{\phi(\xi, 0)}_{\text{on the trajectory}}\exp(-t) = \tanh(\xi)\exp(-t)$

$x/1+t^2$

Ex.2) $\partial u/\partial t + u^2\,\partial u/\partial x = \beta u$ ($\beta > 0$)

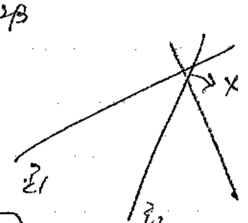
$\Rightarrow \frac{dx}{dt} = u^2 \qquad \underbrace{\frac{du}{dt} = \beta u > 0}_{\text{characteristics.}}$

~~$x = u^2 t + \xi$.~~ $\quad u = F(\xi)\cdot\exp(\beta t)$ — ③

$\frac{dx}{dt} = |F(\xi)|^2\exp(2\beta t) \Rightarrow \int_\xi^x dx = \int_0^t (F(\xi))^2\exp(2\beta\tau)\,d\tau$ (ξ indicates where $t=0$)

$$\Rightarrow x = \frac{(F(\xi))^2\exp(2\beta t)}{2\beta} - \frac{(F(\xi))^2}{2\beta} + \xi.$$

t_{shock}? → $\underline{dx/d\xi = 0}$



→ (x shouldn't change for different trajectories.)

I.C.: $u(x, t=0) = \exp(-x^2)$.

$\Rightarrow$ ③: $u(\xi) = F(\xi)\cdot 1 = \exp(-\xi^2)$ $\quad F(x) = e^{-x^2}$ → continue

$t_s = \frac{1}{2\beta} \ln\left(1 + \frac{\beta \exp(2\xi^2)}{2\xi}\right)$ → multiple t_s for ξ values

⇒ $t_s = \min(t_s)$ ⇒ $\partial t_s / \partial \xi = 0$ → Q) Can you guarantee it's a minimum?

↳ $t_s = \frac{1}{2\beta} \ln(1 + \beta\sqrt{e})$ ⇒ A)

* In $\beta = 0$ case, $u = F(\xi)$ → / / /

$X = (\exp(-2\xi^2))^2 + \xi$ → shock is <u>possible!</u>

① $\frac{dx}{ds} = \frac{B}{A}\frac{dt}{ds}$

for $A(\phi,x,t)\frac{\partial\phi}{\partial t} + B(\phi,x,t)\frac{\partial\phi}{\partial x} + C(\phi,x,t) + D(x,t) = 0$.

② $\frac{d\phi}{ds} = -\frac{C(\phi,x,t)+D(x,t)}{A(\phi,x,t)}\frac{dt}{ds}$

<Confusion in concepts → Eliminated>

. Let's say we have $u_t + 2tx^2 u_x = 0$ $u(x,t=0) = \exp(-x^2)$

characteristic curve $(x(\theta), t(\theta)) \Rightarrow \left.\frac{dx}{d\theta}\right|_\xi = 2tx^2\frac{dt}{d\theta}$ —①

which means that on ξ (trajectory), ① holds.

→ we take $s=\theta=t$ (for convenience only)

$$\Rightarrow \left.dx/dt\right|_\xi = 2tx^2\cdot 1 \Rightarrow \left.\frac{dx}{x^2} = 2t\,dt\right|_\xi \Rightarrow -\frac{1}{x} = t^2+\xi.$$

since we have $-\frac{1}{x} = t^2+\xi$ —②

→ we have $\left.du/dt\right|_\xi = 0$ on trajectory $\Rightarrow u = F(\xi)$

"shouldn't change along characteristic" line.

trajectory

→ changes respect to different ξ

$$\Rightarrow u = F(\xi) = F\left(-\frac{1}{x}-t^2\right)$$

$$u_0(x) = F\left(-\frac{1}{x}-0^2\right) = F\left(-\frac{1}{x}\right) = \exp(-x^2)$$

$$\Rightarrow F(\xi) = \exp\left(-\left(-1/\xi\right)^2\right) = \exp\left(-\frac{1}{\xi^2}\right)$$

$$\Rightarrow u = \exp\left(-\frac{1}{\left(+\frac{1}{x}+t^2\right)^2}\right)$$

Q) what if, we take $-\frac{1}{x}+\xi = t^2$ —③ $\Rightarrow \xi = \frac{1}{x}+t^2$

$$u_0(x) = F\left(\frac{1}{x}\right) = \exp(-x^2) \Rightarrow F(\xi) = \exp\left(-\frac{1}{\xi^2}\right)$$

$$\Rightarrow u = \exp\left(-\frac{1}{\left(\frac{1}{x}+t^2\right)^2}\right)$$

②, ③ are same! //

01/23/2024

Lecture 5 : Wrapping up Characteristic Methods (1st order PDE)

$\phi(x,t) \Rightarrow A\,\partial\varphi/\partial t + B.\,\partial\varphi/\partial x + \tilde{C} = 0.$

$$\Rightarrow \boxed{\frac{dt}{A} = \frac{dx}{B} = \frac{d\varphi}{-\tilde{C}}}$$

Example ;

$$\frac{A}{A}\partial\phi/\partial t + \frac{B}{A}\,\partial\phi/\partial x + \frac{C}{A}\,\partial\phi/\partial y + \boxed{\frac{D}{A}}\frac{\partial\phi}{\partial z} + \frac{E}{A} = 0 \qquad (\phi(x,y,z,t)). \quad — 1$$

$$\Rightarrow d\phi/dt\big|_{z} = \partial\phi/\partial t + \left(\frac{dx}{dt}\right)\Big|_{z}\frac{\partial\phi}{\partial x} + \frac{dy}{dt}\Big|_{z}\frac{\partial\phi}{\partial y} + \boxed{\frac{dz}{dt}}\Big|_{z}\frac{\partial\phi}{\partial z}, \quad — 2$$

(on the characteristics).

Comparing 1 and 2, $\frac{dx}{dt}\Big|_{z} = B/A$, $\frac{dy}{dt}\Big|_{z} = C/A$, $\frac{dz}{dt}\Big|_{z} = D/A.$

Alternatively, $\boxed{\frac{dx}{B} = \frac{dy}{C} = \frac{dz}{D} = \frac{d\phi}{-E}} = \frac{dt}{A}$

- Conservation laws.

$\partial\vec{u}/\partial t + \vec{u}\cdot\nabla\vec{u} = 0$ (primitive form).

$\partial u/\partial t + \underbrace{\frac{d}{dx}\left(\vec{u}^2/2\right)}_{\text{Flux term}} = 0$ (conservative form)

General law : $\underline{\partial u/\partial t + \partial/\partial x\; F(u,x,t) = 0}$

→ Conservation law in integral form always hold

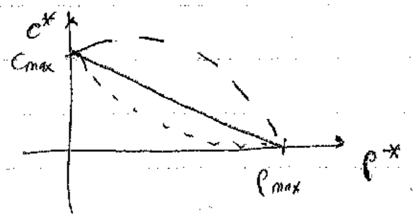
Ex. $\partial\rho^*/\partial t + \partial F^*/\partial x = 0$ (car flow model).

$F(\rho^*) = \rho^*\, c(\rho^*) \rightarrow \partial\rho^*/\partial t + \left(\frac{\partial c}{\partial\rho^*}\rho^* + c^*\right)\partial\rho^*/\partial x = 0.$

$\Rightarrow dx/dt\big|_{z} = \frac{dc^*}{d\rho^*}\rho^* + c^* \longrightarrow g(\rho^*)$

$d\rho^*/dt\big|_{z} = 0 \qquad \rho^* \to$ constant.

Let's use model of



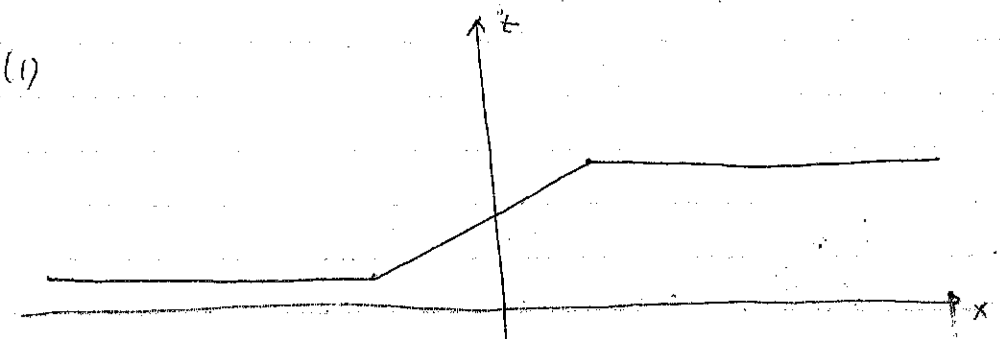
$$\rho = \rho^*/\rho_{max}$$
$$c = c^*/c_{max} \Rightarrow \underline{c(\rho) = 1-\rho}$$

$$c^* = c_{max}\left(1 - \rho^*/\rho_{max}\right)$$

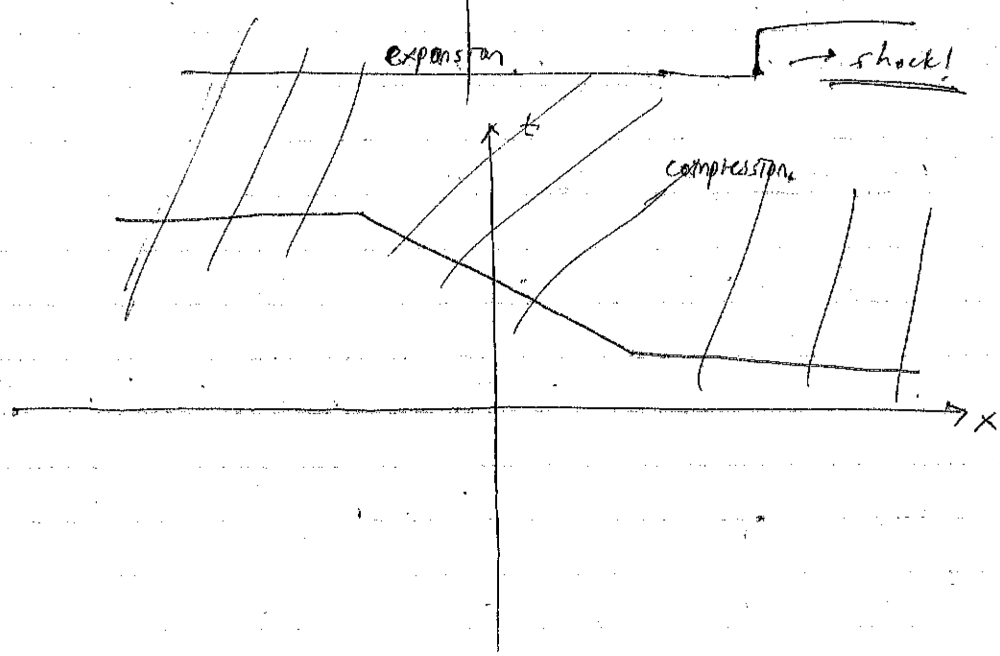
$$\Rightarrow \partial\rho/\partial t + \frac{\partial}{\partial x}\left(\rho c(\rho)\right) = 0 \Rightarrow \partial\rho/\partial t + \left(c + \rho\frac{\partial c}{\partial \rho}\right)\frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \partial\rho/\partial t + (1-2\rho)\,\partial\rho/\partial x = 0 \Rightarrow \begin{cases} dx/dt|_{ch} = 1-2\rho \\ d\rho/dt|_{ch} = 0 \end{cases}$$

(1)



(2)



Weak solution = smooth + countable jumps.

Let's find "shock speed" → solution on differentiable parts, jump across shock.

- Concentrate on one shock : $X_s(t)$

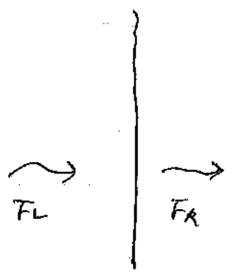
coordinate change : $z = x - X_s(t)$ $\quad \rho(x,t) \rightarrow \rho(z,t)$ $\quad (\tau = t)$

↑ Riding the shock.

$$\partial\rho/\partial t + \partial/\partial x\,(\rho c) = 0 \longrightarrow \partial\rho/\partial\tau + \frac{\partial}{\partial x}\left(\rho c - \rho \dot{X}_s\right) = 0.$$

Inside the car, shock transition is very smooth,
But outside the car, as x evolves. there exists a sudden shock location.

< At frame of reference of shock >



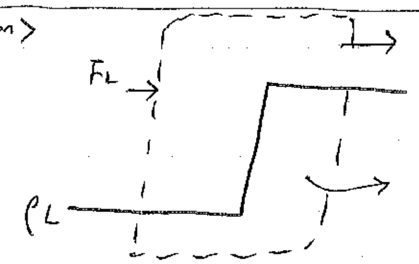
Property of conservation : $F_L = F_R$ Hugoniot condition.

$$\rho_L c_L(\rho) - \rho_L \dot{X}_s = \rho_R c_R(\rho) - \rho_R \dot{X}_s \Rightarrow \dot{X}_s = \frac{\rho_R c_R(\rho_R) - \rho_L c_L(\rho_L)}{\rho_R - \rho_L}$$

< at general frame >.

In general, $\dot{X}_s = \dfrac{F_R - F_L}{\rho_R - \rho_L}$ ← fluxes; ← state variables.

< Intuition >

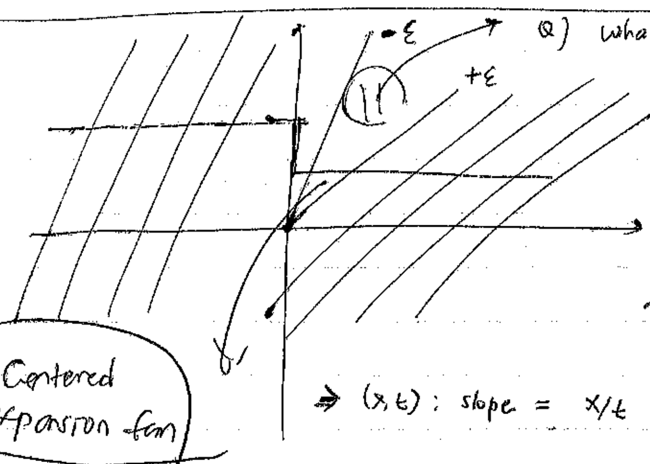


$$\int_{cv} \rho(t+\Delta t)\,dx - \int_{cv} \rho(t)\,dx = \dot{X}_s\,\Delta t(\rho_L - \rho_R)$$

$\parallel$ Integral of fluxes in time domain

$(F_L - F_R)\Delta t$

Q) what happens here?



$$\left.dx/dt\right|_z = 1 - 2\rho \qquad \left.d\rho/dt\right|_z = 0$$
→ Straight Characteristic lines.

∴  is a solution.

Centered expansion fan

→ (x,t) : slope $= x/t$ ($\because$ passes through origin) $= 1 - 2\rho$ → $\rho = \dfrac{1 - x/t}{2}$

01/25/2024.

Lecture 6. (ch3).

Second order PDE. (ex. wave equation).

$\partial^2 u/\partial t^2 - c_0^2\, \partial^2 u/\partial x^2 = 0$ (we need two b.c., two init. cond.)

↳ D'Alembert Soln.

$$\left.\begin{aligned} u(x,t=0) &= f(x) \\ \frac{\partial u}{\partial t}(x,t=0) &= g(x) \end{aligned}\right\} \Rightarrow u(x,t) = \frac{1}{2}\left\{ f(x-c_0t) + f(x+c_0t) \right\} + \frac{1}{2c_0}\int_{x-c_0t}^{x+c_0t} g(x')\,dx' \quad \text{— ①}$$

(Note: $\partial^2 u/\partial t^2 - c_0^2 \nabla^2 u = 0$
Linear PDE → { Transformation method, Eigenfunction expansions })

proof of ①) Method of characteristics after convert to 1st order PDEs.

Approach : 1) Rewrite system into coupled 1st-order PDEs
2) Decouple them!
3) Solve individually.
4) Construct back solution for original PDE.

- $u_1 = \partial u/\partial x \quad u_2 = \partial u/\partial t$

$$\vec{V} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \partial/\partial t\,(\partial u/\partial t) - c_0^2\,\partial/\partial x\,(\partial u/\partial x) = 0$$

$$\Rightarrow \partial u_2/\partial t - c_0^2\, \partial u_1/\partial x = 0 \quad \text{—— (1)}$$

$$\partial u_1/\partial t = \partial/\partial t\,(\partial/\partial x\, u) = \partial/\partial x\left(\frac{\partial u}{\partial t}\right) = \partial/\partial x\, u_2 \quad \text{— (2)}$$

(1), (2) becomes

$$\begin{aligned} \partial u_1/\partial t - \partial u_2/\partial x &= 0 \\ \partial/\partial t\,(u_2) - \cancel{c_0^2}\,\partial/\partial x\,(u_1) &= 0 \end{aligned} \Rightarrow \frac{\partial \vec{V}}{\partial t} + \underbrace{\begin{pmatrix} 0 & -1 \\ -c_0^2 & 0 \end{pmatrix}}_{=B} \frac{\partial \vec{V}}{\partial x} = \vec{0}.$$

$$\Rightarrow \frac{d\vec{U}}{dt} - \begin{pmatrix} 0 & 1 \\ c_0^2 & 0 \end{pmatrix} \frac{d\vec{U}}{dx} = \vec{0}$$

↓ Diagonalization

⇒ B ⇒ Eig. val. decomp. : $\begin{pmatrix} 1 \\ -c_0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ c_0 \end{pmatrix}$ → $Q = \begin{pmatrix} 1 & 1 \\ -c_0 & c_0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix}$

⇒ $B = Q\Lambda Q^{-1}$ whereas $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = Q^{-1} U$ (we defined it)

$$\frac{\partial \vec{U}}{\partial t} + Q\Lambda Q^{-1}\frac{\partial \vec{U}}{\partial x} = 0 \rightarrow Q^{-1}\frac{\partial \vec{U}}{\partial t} + \Lambda Q^{-1}\frac{\partial \vec{U}}{\partial x} = 0$$

$$\Rightarrow \frac{\partial \vec{V}}{\partial t} + \Lambda \frac{\partial \vec{V}}{\partial x} = 0$$

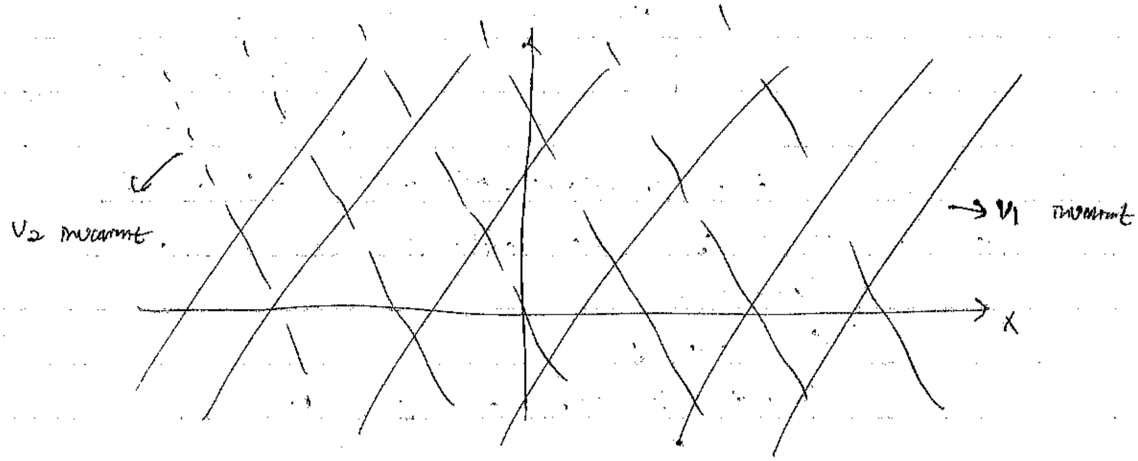
→ Now it's decoupled!

$$\Rightarrow \frac{\partial}{\partial t}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \left.\begin{aligned} \partial V_1/\partial t + \lambda_1\, \partial V_1/\partial x &= 0 \\ \partial V_1/\partial t + \lambda_2\, \partial V_2/\partial x &= 0 \end{aligned}\right\}$$

Characteristic Variables

<Decoupled systems>



I.C. for V_1 and V_2. $\quad u(x,t=0) = f(x)$,

$$\frac{\partial u}{\partial t}(x,t=0) = 0$$

① $\left.\begin{aligned} V_1(x,t=0) &= f'(x)/2 \\ V_2(x,t=0) &= f'(x)/2 \end{aligned}\right\}$

$$V_1(x,t) = F^+(\xi^+) = F^+(x - c_0 t)$$
$$V_1(x,t=0) = F^+(x) = \tfrac{1}{2} f'(x)$$
$$\checkmark\ V_1(x,t) = \tfrac{1}{2} f'(x - c_0 t)$$

$$V_2(x,t) = F^-(x + c_0 t)$$
$$V_2(x,t=0) = F^-(x) = \tfrac{1}{2} f'(x)$$
$$\checkmark\ V_2(x,t) = \tfrac{1}{2} f'(x + c_0 t)$$

② $U = Q\vec{V} \quad \Rightarrow \quad u_1 = V_1 + V_2 \qquad u_1 = \frac{1}{2}\left[f'(x - c_0 t) + f'(x + c_0 t)\right]$

$\qquad\qquad\qquad\quad u_2 = c_0(V_2 - V_1) \qquad \| \ du/dx$

$$\Rightarrow u = \int du/dx = \frac{1}{2}\left\{f(x - c_0 t) + f(x + c_0 t)\right\} + I(t)$$

③ From ②, $u(x,t) = \frac{1}{2}\{f(x-c_0t) + f(x+c_0t)\} + I(t)$.

We can use initial condition of $\partial u/\partial t\ (x, t=0)$

$u(x,t=0) = f(x) + I(0) = f(x) \Rightarrow I(0) = 0$

$\cancel{\partial u/\partial t\ (x,t)}\Big|_{t=0} = \frac{1}{2}\{-c_0 \cdot f(x-c_0t) + c_0 f(x+c_0t)\}\Big|_{t=0} + \partial I/\partial t\Big|_{t=0}$

$\Rightarrow\quad 0 = \frac{1}{2}\{\cancel{-c_0f(x) + c_0f(x)}\} + \left(dI/dt\Big|_{t=0} = 0\right)$ —— (5)

Bach

But if we defined init. cond. to be $\partial u/\partial t\ (x, t=0) = g(x)$,

$g(x) = dI/dt\Big|_{t=0}$ is satisfied.

Let's talk about (5) case,

plug in $u = \frac{1}{2}(f(x-c_0t) + f(x+c_0t)) + I$ to

$\partial^2u/\partial t^2 - c_0^2\, \partial^2u/\partial x^2 = 0$

$\Rightarrow\ 0 - 0 + I'' - c_0^2 I = 0 \Rightarrow I(0) = 0 \rightarrow I = 0$ //

$I'(0) = 0$.

$I''(0) = 0$

< Generalize >

$$\partial^2u/\partial x^2 + \frac{B}{A}\frac{\partial^2 u}{\partial x\partial y} + \frac{C}{A}\frac{\partial^2 u}{\partial y^2} = \frac{D}{A} \rightarrow u_1 = \partial u/\partial x,\ u_2 = \partial u/\partial y$$

$$\underbrace{\frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} B/A & C/A \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} D/A \\ 0 \end{pmatrix}}_{\mathbb{B}} \qquad \partial u_2/\partial x = \partial u_1/\partial y \ (\text{2nd equation})$$

can $\mathbb{B}$ matrix diagonalized → if so, we can easily solve!

Q) Determine diagonalizability.

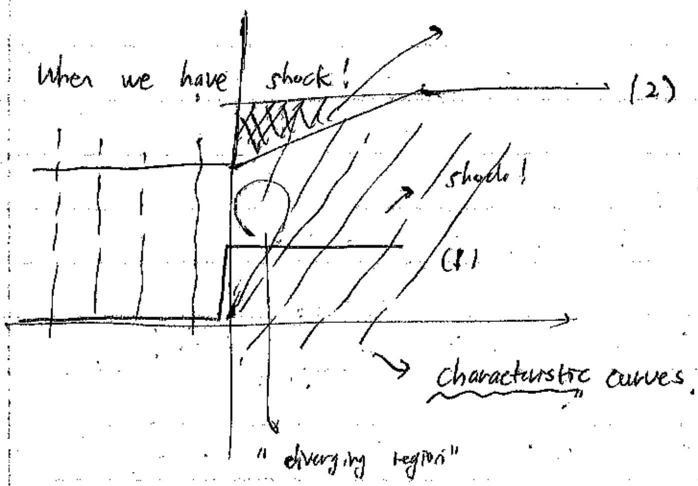
$\rightarrow |\lambda I - \mathbb{B}| \Rightarrow \boxed{\Delta = B^2 - 4AC}$ (i) >0 hyperbolic (two distinct eigvals).

GR

(ii) $=0$ parabolic (repeated)

(iii) <0. elliptic. (two complex eigvals

When we have shock!



(1) becomes (evolves to) (2).

01/30/2024.

Lecture. 7. (Last lecture of method of characteristics).

- Wave eqn.

$u(x,t)$ where $\{(x,t) \mid -\infty < x < \infty, \ 0 \le t < \infty\}$

$u(x,t)$ satisfies $\partial^2 u/\partial t^2 - c_0^2 \, \partial^2 u/\partial x^2 = 0$ $\begin{cases} u(x, t=0) = f(x) \\ \dot{u}(x, t=0) = g(x) \end{cases}$

D'Alembert sol. $u(x,t) = \frac{1}{2}\{\underbrace{f(x-c_0 t)}_{\text{Right going wave}} + \underbrace{f(x+c_0 t)}_{\text{Left going wave}}\} + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} g(\tau)\,d\tau.$

(1) New way to solve! $\equiv$ $\boxed{\text{Operators!}}$

$$\underbrace{\left(\partial/\partial t + c_0 \, \partial/\partial x\right)}_{\text{Left}}\underbrace{\left(\partial/\partial t - c_0 \, \partial/\partial x\right)}_{\text{Right}} u = 0 \quad \equiv \text{Double advection.}$$

$\Rightarrow$ Advection operators.

(i) $\xi^+ = x - c_0 t$
(ii) $\xi^- = x + c_0 t$

} Trajectories where information is transferred.

Change (transform) $(x,t) \to (\xi^+, \xi^-)$ $\Rightarrow \frac{\partial}{\partial \xi^+}\frac{\partial}{\partial \xi^-} u = 0. \Leftrightarrow u_{\xi^+\xi^-} = 0.$

$\Rightarrow$ we know $u(\xi^+, \xi^-) = G^+(\xi^+) + G^-(\xi^-)$
$= G^+(x - c_0 t) + G^-(x + c_0 t)$

Using B.C. where $u(x, t=0) = G^+(x) + G^-(x) = f(x)$
$\frac{\partial}{\partial t}u(x, t=0) = -c_0 G^{+\prime}(x) + c_0 G^{-\prime}(x) = g(x)$ } —(*)

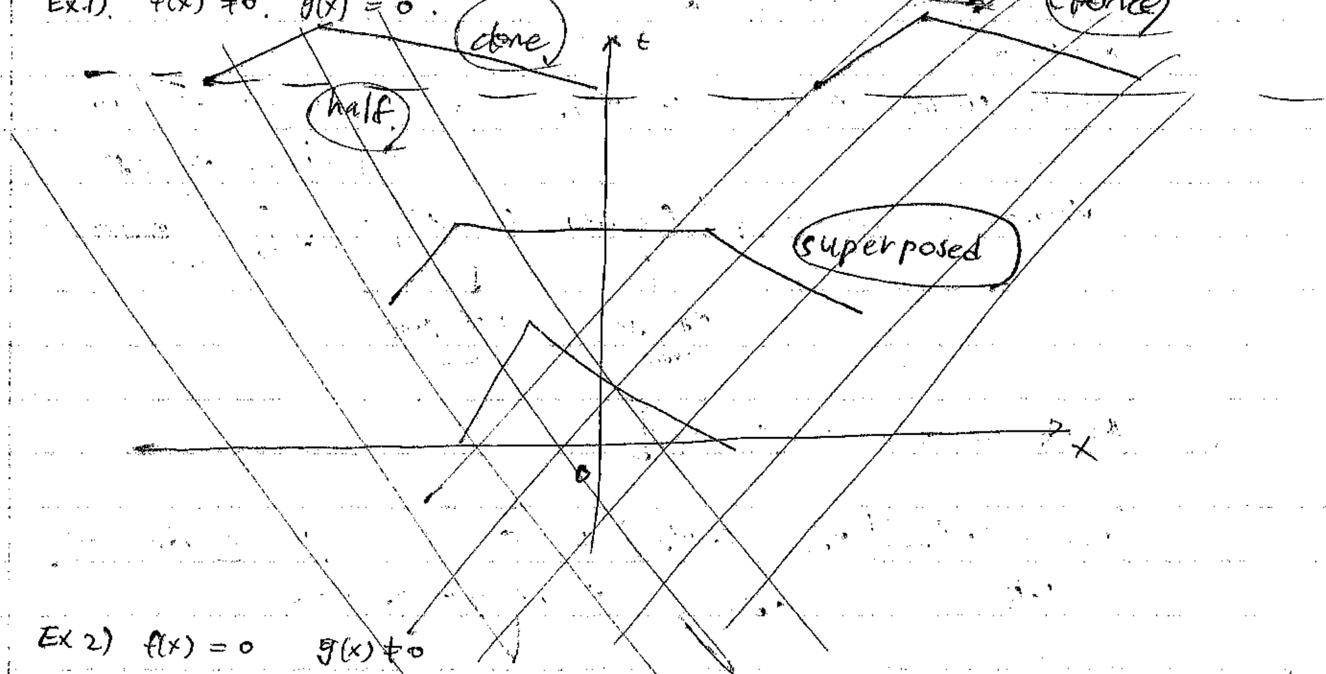
(*) becomes. $G^+(x) + G^-(x) = f(x)$
$G^+(x) - G^-(x) = -\frac{1}{c_0}\int g(\tau)\,d\tau.$

$\Rightarrow G^+(x) = \dfrac{f(x) - \frac{1}{c_0}\int g(\tau)d\tau}{2}$ and $G^-(x) = \dfrac{f(x) + \frac{1}{c_0}\int g(\tau)d\tau}{2}.$

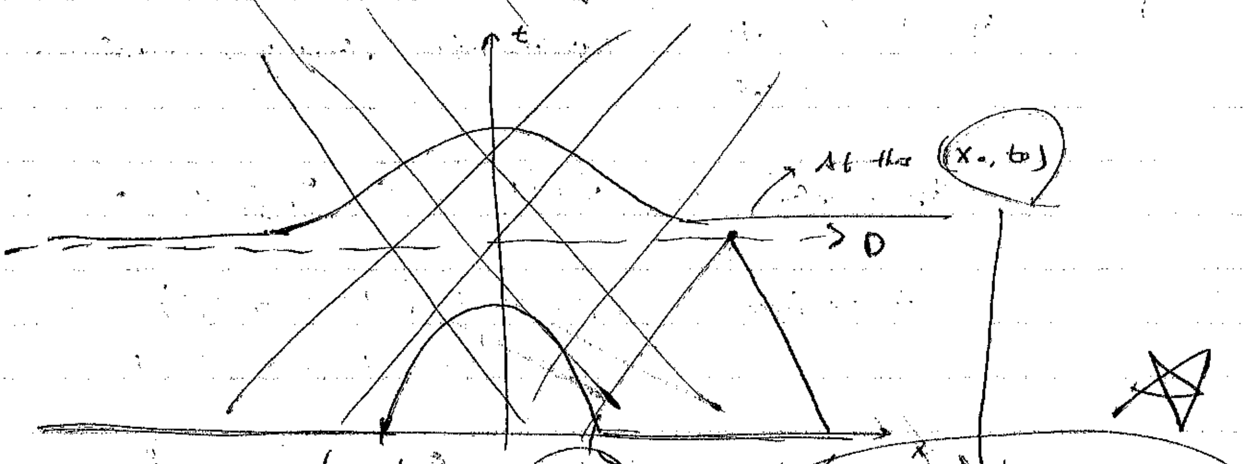
$$\Rightarrow u = G^+(x-c_0t) + G^-(x+c_0t) = \frac{1}{2}\{f(x-c_0t) + f(x+c_0t)\} + \frac{1}{2c_0}\int_{x-c_0t}^{x+c_0t} g(\tau)\,d\tau \quad \#$$

- Interpretations of solutions.

Ex.1) $f(x) \neq 0$. $g(x) = 0$.



Ex 2) $f(x) = 0$ $g(x) \neq 0$



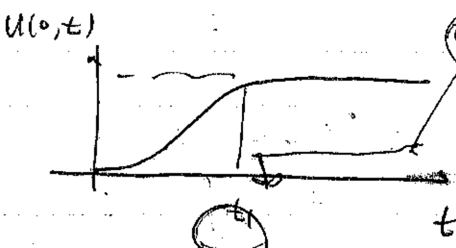
$[x_0 - c_0 t_0 ,\ x_0 + c_0 t_0]$

"Domain of influence"

Each point on D, $u(x,t) = \int_{x-c_0 t}^{x+c_0 t} g(\tau)\, d\tau$

As t grows, $[x - c_0 t,\ x + c_0 t]$ gets larger, thus

the shape ⌒ → ⌒ ⇒ ⌒ Expands!



Example 3). One way waves. (wave actuators, wave absorbers, b.c.s).

$u(x,t) = f(x - c_0 t)$ is a solution. $\Longleftarrow$ $\left\{ \begin{array}{l} \text{I.C.} \quad u(x,0) = f(x) \\ \qquad u'(x,0) = -c_0 f(x) \end{array} \right.$

$\partial u/\partial t = -c_0 f'(x - c_0 t)$

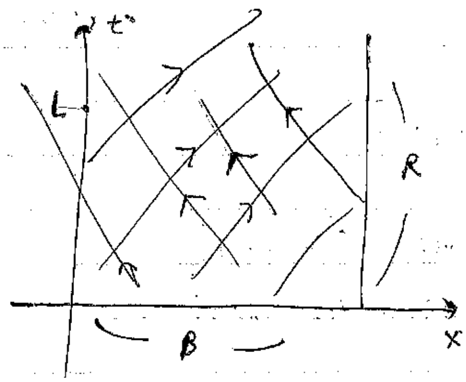
$\vec{v}_1 = \frac{1}{2}\left(\partial u/\partial x - \frac{1}{c_0}\frac{\partial u}{\partial t}\right) = f'$

$\vec{v}_2 = \frac{1}{2}\left(\partial u/\partial x + \frac{1}{c_0}\frac{\partial u}{\partial t}\right) = 0$ $\quad v_2 = 0$ at $t \geq 0$

↳ " carries no information)!

- Initial conditions & Boundary conditions.

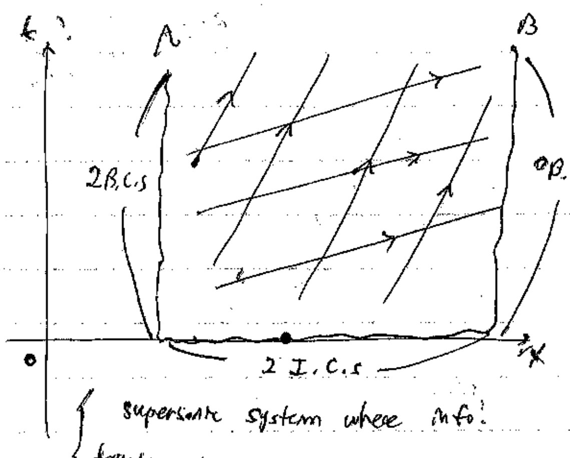
Q) where? How many? what? → well-posed problem.



$\left\{ \begin{array}{l} B: \text{two I.C.s} \quad \text{— when } t = 0 \\ R: \text{one B.C.s} \\ \text{point } L: \text{one. B.C.s} \end{array} \right\}$ — where it's coming from.

In general, you need

Edge conditions specify the characteristic that are coming INTO the domain of interest (from outside)



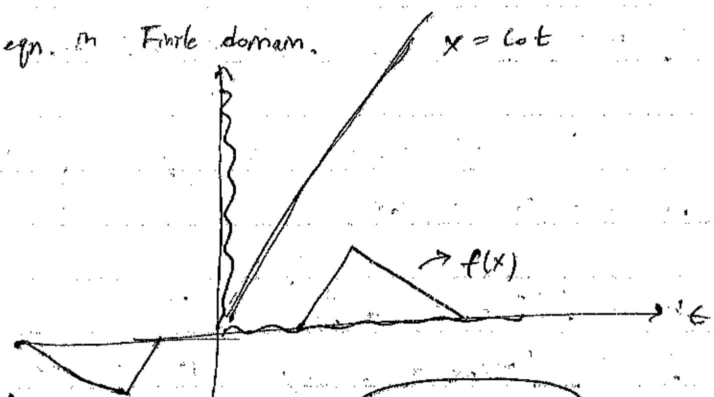
☆

comes in only at edge A, not B.

{ supersonic system where info travels at speed of sound relative to flow.

Solution to wave eqn. in Finite domain.

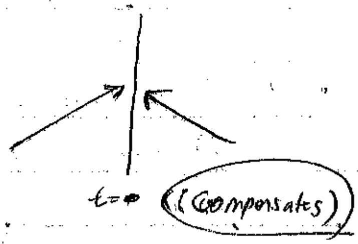
$$\begin{pmatrix} 0 \le x < \infty \\ 0 \le t < \infty \end{pmatrix}$$



case 1) $u(x,t)\big|_{x=0} = 0 \rightarrow$ Dirichlet B.C.

"Method of images" $\equiv$ Embed into a problem $-\infty < x < \infty$

$$\tilde{f}(x, t=0) = \begin{cases} f(x) & 0 \le x < \infty \\ -f(-x) & -\infty < x \le 0 \end{cases}$$

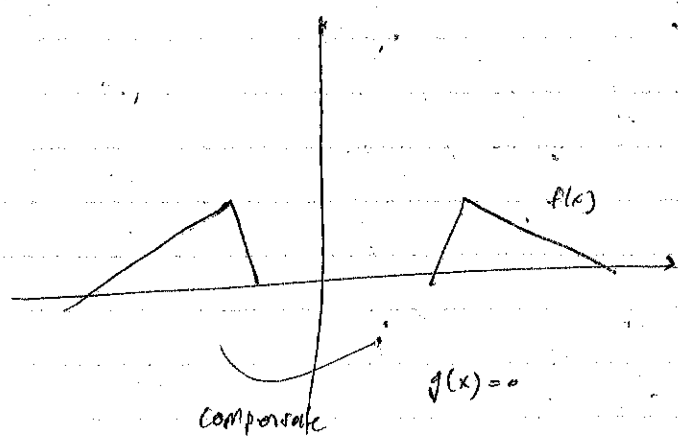


$$u(x,t) = \frac{1}{2}\left\{ \tilde{f}(x - c_0 t) + \tilde{f}(x + c_0 t) \right\}$$

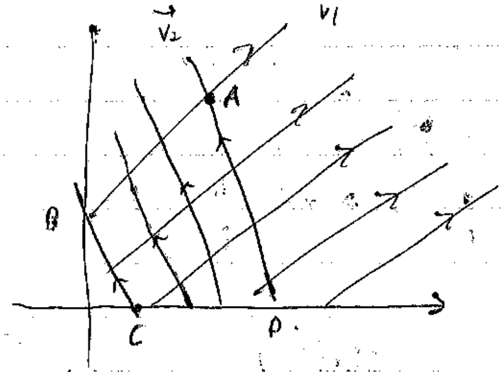
$$\begin{cases} x > c_0 t \rightarrow u = \frac{1}{2}\left\{ f(x + c_0 t) - f(x - c_0 t) \right\} \\ x < c_0 t \rightarrow u = \frac{1}{2}\left\{ f(x + c_0 t) + f(x - c_0 t) \right\} \end{cases}$$

Case 2) $\partial u / \partial x \, (x = 0, t) = 0 \rightarrow$ Neumann B.C.

$$\tilde{f}(x) = \begin{cases} f(x) & 0 \le x < \infty \\ f(-x) & -\infty < x < 0 \end{cases}$$



• Time dependent B.C.s.

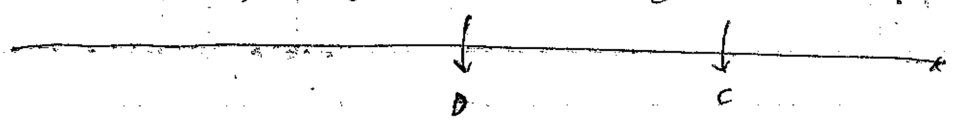


$u(x=0,t) = h(t) \rightarrow u(A)$ in terms of $u(C)$ and $u(D)$, $h(t_A)$

V_1 is constant along $\vec{AB}$ + V_2 is constant along $\vec{BC}$

$u(A) = u(x_A, t_A)$

$= h(t_A - x_A/c_0) + \frac{1}{2} f(x_A + c_0 t_A) - \frac{1}{2} f(c_0 t_A - x_A)$

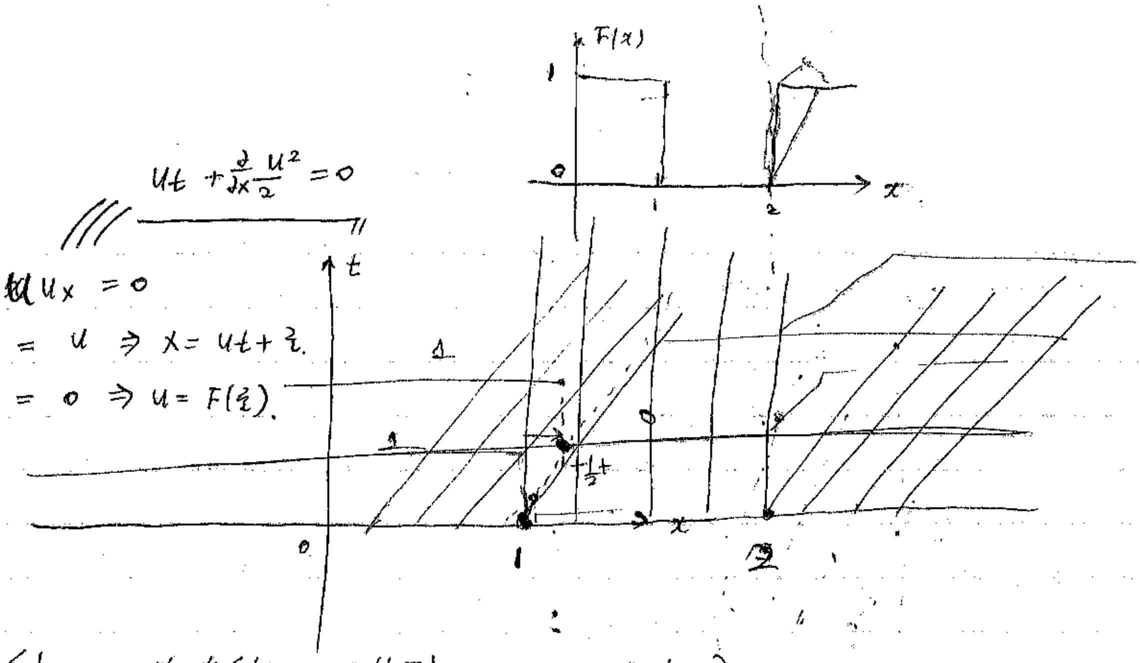


$u_t + \frac{\partial}{\partial x}\frac{u^2}{2} = 0$

Q) $u_t + u u_x = 0$

$dx/dt = u \Rightarrow x = ut + \xi$

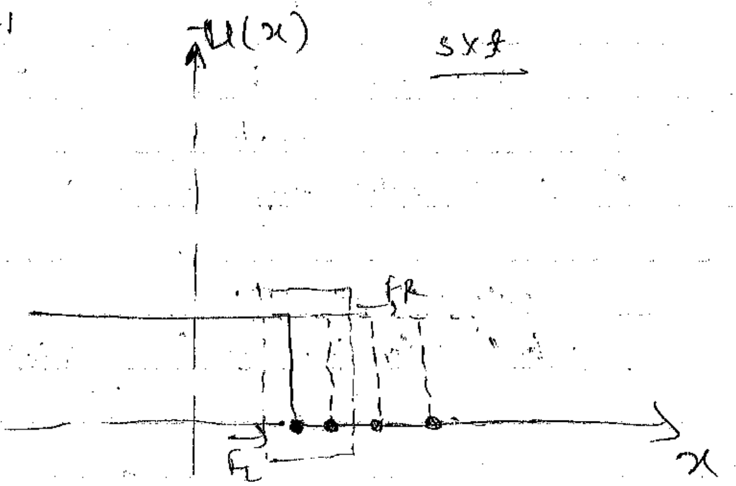
$du/dt = 0 \Rightarrow u = F(\xi)$



$\xi < 1$ $\quad x - t < 1 \quad \to u = 1$

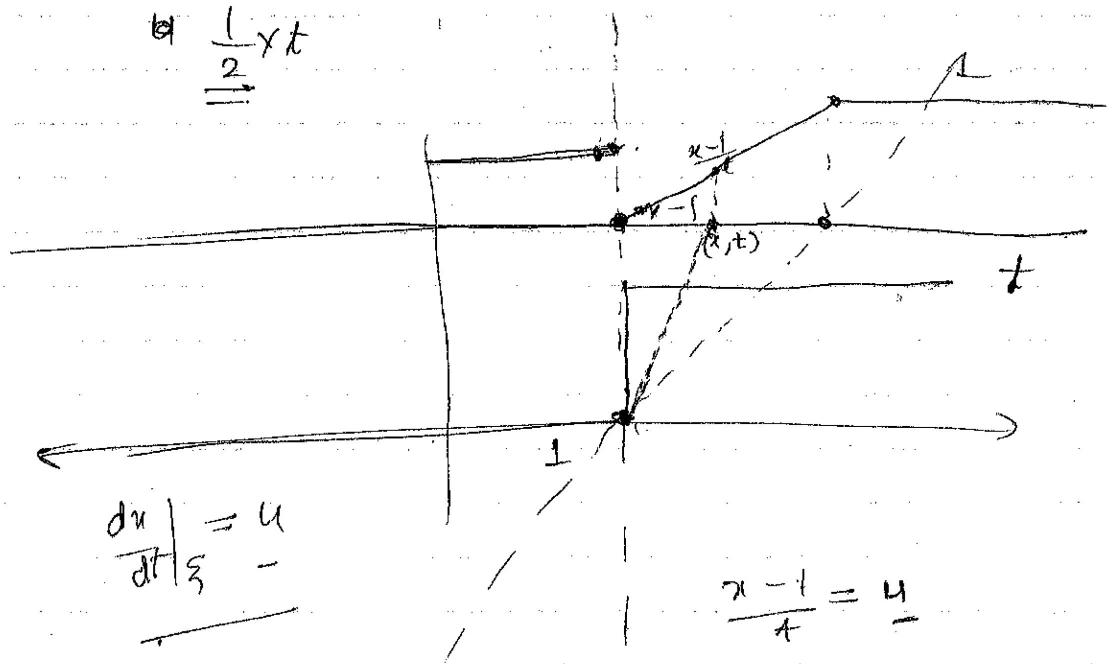
$1 < \xi < 2$ $\quad 1 < x < 2 \quad \to u = 0$

$\xi > 2$ $\quad x - t > 2$



$\boxed{\frac{U_R + U_L}{2}}$

$\frac{1}{2} \times t$



$\left.\frac{dx}{dt}\right|_{\xi} = u$

$\frac{x-1}{t} = u$

02/01/2024

Lecture 8. Seperation of variables.

$\partial^2 u/\partial t^2 - c_0^2 \, \partial^2 u/\partial x^2 = 0 \rightarrow dx/dt = \pm c_0 \quad \begin{cases} \xi^+ = x - c_0 t \rightarrow \xi \\ \xi^- = x + c_0 t \rightarrow \eta \end{cases}$

$\rightarrow \underline{B} = \begin{bmatrix} 0 & -1 \\ -c_0^2 & 0 \end{bmatrix}$ where $|\underline{B} - \lambda \underline{I}| = 0$.

general theory

In general, $A\, u_{xx} + B\, u_{yx} + C\, u_{yy} = D \Rightarrow \partial y/\partial x = \frac{B}{2A} \pm \sqrt{B^2 - 4AC}/2A$

wave equation is identical to $\frac{\partial u}{\partial \xi \partial \eta} = 0 \Rightarrow u_{\xi\eta} = 0$

Ex) Heat equation : $\partial u/\partial t = \partial^2 u/\partial x^2 \quad (0 < x < 1,\ t > 0)$

B.C. : $u(0,t) = u(1,t) = 0 \quad t > 0$

I.C. : $u(x,0) = f(x) \qquad f(x) \in C^2(0,1)$ (Class of functions differentiable twice).

Ansatz ; $u(x,t) = \phi(t)\, \psi(x)$.

if $u(x,t) \neq 0 \Rightarrow \phi, \psi$ are all non-zeros $\Rightarrow \frac{\phi'}{\phi}(t) = \frac{\psi''}{\psi}(x) = \text{constant}$

$\Rightarrow \frac{\phi'}{\phi}(t) = \frac{\psi''}{\psi}(x) = -\lambda$ (because it should not diverge as $t \to \infty$)

$(\Rightarrow \lambda > 0)$

$\Rightarrow \phi = A\exp(-\lambda t) \qquad \psi = B\cos\sqrt{\lambda}x + C\sin\sqrt{\lambda}x$

B.C. $\begin{cases} \psi(0) = \psi(1) = 0 \Rightarrow B = 0,\ \sqrt{\lambda} = \pi. \Rightarrow \lambda = \pi^2 \Rightarrow \underline{\psi(x) = C\sin(\pi x)} \\ \phi(t) = A\exp(-\pi^2 t) \end{cases}$

$\Rightarrow \quad u(x,0) = C\sin(\pi x)\cdot A = f(x). \Rightarrow AC = f(x)/\sin(\pi x)$ for $\forall x \in [0,1]$.

$\Rightarrow \underbrace{u(x,t) = \left(\frac{f(x)}{\sin \pi x}\right) \exp(-\pi^2 t)\sin(\pi x)}$, (X) wrong!

constant.

we get. $\psi(x) = C_k \sin(k\pi x) \qquad \phi(t) = A\exp(-k^2\pi^2 t)$.

$\Rightarrow u_k(x,t) = C_k \cdot \sin(k\pi x)\exp(-k^2\pi^2 t)$.

$\sum_{k=0}^{\infty} u_k(x,0) = f(x) \Rightarrow \sum_{k=0}^{\infty} C_k \sin(k\pi x) \cdot 1 = f(x)$.

Q) How to guarantee $u(x,t) = \sum_{k=1}^{\infty} C_k \cdot u_k(x,t)$ is finite?

$\Rightarrow$ C_k must rapidly go to zero.

[Easy way]

$$u(x,t)\Big|_{t=0} = \sum_{k=1}^{\infty} c_k \sin(k\pi x)\exp(-k^2\pi^2 t)\Big|_{t=0} = \sum_{k=1}^{\infty} C_k \sin(k\pi x) = f(x)$$

$$\Rightarrow \int_0^1 \sum_{k=1}^{\infty} C_k \sin(k\pi x) \sin(p\pi x)\, dx = \int_0^1 f(x)\sin p\pi x\, dx$$

$$\begin{cases} C_k \int_0^1 \sin^2 p\pi x\, dx & (k=p) \\ 0 & (k\neq p)\end{cases}$$

$$= \int_0^1 f(x)\sin p x\, dx$$

$$C_k \int_0^1 \frac{1}{2}(1-\cos 2p\pi x)\, dx = \frac{C_k}{2}$$

$$\therefore \; C_k = 2\int_0^1 f(x)\sin(k\pi x)\, dx \rightarrow u(x,t) = \sum_{k=1}^{\infty} C_k u_k(x,t) \quad \#$$

- TA Session :

01/02/2024.

Q) Why it not work for point B.C.s.? → Code wrong?

$$\psi_1(x,t) = \frac{1}{2} f(x-c_0t) + \frac{1}{2\rho_0 c_0} g(x-c_0t)$$

ψ_2 " = " " − " " +

if $\psi_2 = 0$, we only have "right going waves" in the solution.

• Method of images.

Wave equation on semi-infinite domain.

$$u_{tt} - c_0^2 u_{xx} = 0. \quad (x \in \mathbb{R}\ -\infty \le x < +\infty,\ t \ge 0)$$

B.C.s : $\partial u / \partial x\,(0,t) = 0$.

I.C.s : $u(x,0) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 0 & 2 \le x < \infty \end{cases}$

$\partial u / \partial t\,(x,0) = 0.$

* Method of images tries to satisfy the boundary condition.

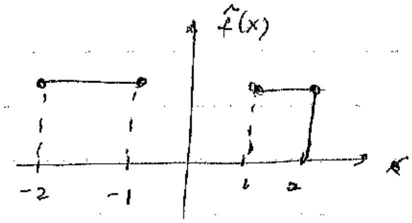
• we presume $\tilde{u}(x,t)$ where.

$$\partial^2 \tilde{u} / \partial t^2 - c^2 \partial^2 \tilde{u} / \partial x^2 = 0. \quad ((-\infty \le x < \infty).\ (t \ge 0))$$

I.C.: $\tilde{u}(x,0) = \tilde{f}(x)$ → we design this!

$\partial \tilde{u} / \partial t\,(x,0) = 0.$

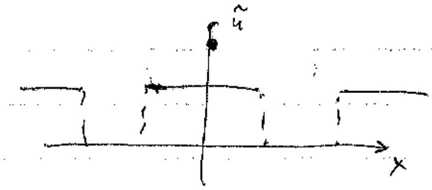
$$\tilde{f}(x) = \begin{cases} f(x) & (\infty > x > 0) \\ f(-x) & (-\infty < x \le 0) \end{cases} \Rightarrow$$



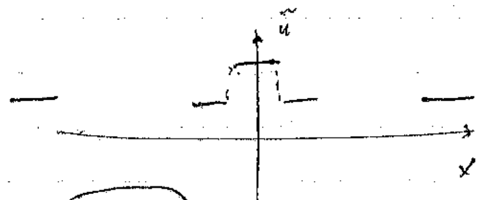
(even extension of f)

$$\tilde{u}(x,t) = \frac{1}{2} \tilde{f}(x - ct) + \frac{1}{2} \tilde{f}(x + ct). \quad (\text{by d'Alembert's method})$$

↳ $\tilde{u}$ is



$t = 0$



$t > 0$

→ $\tilde{u}$ has $\partial / \partial x\ \tilde{u}(0,t) = 0$ → satisfied!

02/06/2024.

Lecture. 9.

- Weak derivatives.

infinitely differentiable.

Let $C_c^{\infty}(U)$ a space, compact, infinitely differentiable and on domain U

↳ compact

→ Define $\phi : U \to \mathbb{R}$. Now, consider function $u \in C^1(U)$

$$\int_U u \frac{\partial \phi}{\partial x_i} dx = -\int_U \frac{\partial u}{\partial x_i}\phi \, dx + \cancel{B}^{\nearrow 0} \quad (\because \text{ compact space} \to B = 0)$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \ \underline{u\phi}$

, for $i = 1, 2, \dots, n$ dimensions

Q) what if $u \notin C^1(U)$ → still make sense

Ø ∵ find V function that is locally summable. $V = \partial u / \partial x_i$

such that $\int_U u \, \partial\phi/\partial x_i = -\int V \phi \, dx$

⇒ Integral allows you to not care about u function's differentiability.

⟨Generalization⟩

→ localized.

Suppose $u, v \in L^1_{loc}(U)$ and α is multi-index. $\quad \int |f(x)| dx < \infty \equiv f \in L^1_{loc}$

We say. V is the α^{th} weak derivative of s/t

$V = D^{\alpha} u \Rightarrow D^{\alpha} u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$ provided;

$$\int_U u D^{\alpha}\phi \, dx = (-1)^{|\alpha|}\int_U V\phi \, dx, \quad \forall \phi \in C_c^{\infty}(U)$$

E.g.) $u(x) = \begin{cases} x & 0 < x \le 1 \\ 1 & 1 < x \le 2 \end{cases} \qquad v(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases}$

Ø $u' = V$ except for $x = 1$ → Doesn't make sense.

However,

$$\int_0^2 u \phi' dx = + \int_0^1 x\phi' dx + \int_1^2 \phi' dx$$

$$= -\int_0^1 \phi dx + \phi(1) - \cancel{\phi(0)}^{\nearrow 0} + \cancel{\phi(2)}^{\nearrow 0} - \phi(1)$$

$(\because bc)$

$$= -\int_0^1 \phi dx + \underline{\phi(1) - \phi(1)} = \underline{-\int_0^2 V\phi dx} \quad \#$$

Q) Does it converge?

$\phi(1+) - \phi(1-) = ?$

◦ Sobolev space.

Def) $\boxed{W^{k,p}(U)}$ → Sobolev spaces.

$u: U \to \mathbb{R}$ s/t for each α, $|\alpha| \le k$

1) $D^{\alpha}(u)$ exists (k time differentiable) in the weak sense.

2) and belongs to $L^p(U) \equiv \left(\int_U |f(x)|^p\right)^{1/p} < \infty$

• A special case ($p=2$)

$W^{k,2}(U) = H^k(U)$ s/t $\left[\int_U |f(x)|^2 dx\right]^{1/2} < \infty$ $= \|f\|_2$

• $H^0(U) = L^2(U)$ s/t L_2 norm comes here. (l_2 norm is for finite matrix.)

Def) If $u \in H^k(U)$

$$\|u\|_{|\alpha|^k} = \left[\sum_{|\alpha| \le k} \left(\int_U |D^\alpha u|^p dx\right)^{1/p}\right]$$

Example: $\|u\|_{H^2} = \left(\int_U |u|^2 dx\right)^{1/2} + \left(\int_U |\partial u/\partial x|^2 dx\right)^{1/2} + \left(\int \left|\frac{\partial^2 u}{\partial x^2}\right|^2 dx\right)^{1/2} < \infty$

• Properties of Hilbert spaces.

For every $f, g \in H$ we can define a scalar (f,g)

(f,g) scalar has

(1) $(f,g) \ge 0$ $\forall f,g \in H$

(2) $(f,f) = 0$ iff $f = 0$

(3) $(\lambda f, g) = \lambda (f,g)$

(4) $(f,g) = (g,f)$

(5) $(f+g, h) = (f,h) + (g,h)$

Eg) $L^2(\Omega)$ with $\Omega \in \mathbb{R}^N$

$(f,g) = \int_\Omega f(x)\,\overline{g(x)}\,dx = \langle f|g \rangle$.

Eg. 2) $L^2(\Omega)$ with $\Omega \in \mathbb{R}^N$

$(f,g) = \int_\Omega f(x) g(x) w(x)\,dx$ $\quad w(x) > 0$

Thm

For H being Hilbert space, if we set $\|f\| = \sqrt{(f,f)}$ then $\|f\|$ is a norm

~~Thm~~ Orthogonality

$\langle f, g\rangle = 0 \Leftrightarrow f, g$ are orthogonal. (iff) $f, g \in H$

Q) can we define orthogonality for $\|f\| = 0$?

Ex.) $f_n(x) = \sin(nx)$ $\{f_n\}_{n=1}^{\infty}$ is orthogonal in L^2

Infinite orthogonal sequences.

- Equipped with project operator P_E
- H $\{f_n\}_{n=1}^{\infty}$ is a countable orthogonal set in H if $f_n \neq 0$ for the

simply $n \to \infty$, $H = \text{span}(\{f_n\}_{n=1}^{\infty})$

then, $\underline{P_E\, g = \sum_{n=1}^{\infty} \frac{(g, f_n)}{(f_n, f_n)} f_n}$ (projection)

↳ (If this series converges)

$\Rightarrow$ let $e_n = f_n / \|f_n\|$, $c_n = (g, e_n)$, $E_N = \text{span}\{f_1 \sim f_N\}$

such that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal set.

$$\Rightarrow P_E\, g = \sum_{n=1}^{N} c_n e_n$$

note Bessel's inequality

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |(g, e_n)|^2 \leq \|g\|^2 \quad \therefore \lim_{n\to\infty} c_n = 0$$

$\|g\| < \infty$

- Infinite orthogonal sequences.

Riesz - Fischer Thm

$\sum_{n=1}^{\infty} c_n e_n \Rightarrow$ is convergent to $P_E g \equiv g$

02/08/2024.

Lecture 10 – Seperation of Variables. – operators.

summarize all the materials in lecture 9,

Thm.

let $\{e_n\}_{n=1}^{\infty} = f_n / \|f_n\|$ be an orthonormal set in H.

a) $\{e_n\}_{n=1}^{\infty}$ form a basis in H

b) $\underbrace{g = \sum_{n=1}^{\infty} (g, e_n) e_n}_{=\text{ Generalized Fourier series.}}$ for $\forall g \in H$ * we can define abitrary inner products $(\cdot, *)$

c) $\|g\|^2 = \sum_{n=1}^{\infty} |(g, e_n)|^2 \rightarrow$ Bessel's equality.

d) $\{e_n\}_{n=1}^{\infty}$ is complete in H.

• Operators (bounded linear operators).

$$B(X,Y) = \left\{ \begin{matrix} T : X \rightarrow Y \\ \text{is linear} \end{matrix} \right\} \quad \|T\|_{XY} < \infty$$

$$\left(\text{where} \quad \|T\|_{XY} = \sup_{f \neq 0} \frac{\|Tf\|_Y}{\|f\|_X} \quad \equiv \text{"maximum is bounded."} \right)$$

let $\mathcal{L}u = a_2(x) u'' + a_1(x) u' + a_0(x) u$ on $[a,b]$ domain.

"a general 2nd order diff. operator (op.)." $\left(\begin{matrix} a_j(x) \in C([a,b]) \\ \text{and } a_2(x) \neq 0 \end{matrix} \right)$

$$\text{B.C.:} \quad \begin{aligned} B_1 u &= c_1 u(a) + c_2 u'(a) \\ B_2 u &= c_3 u(b) + c_4 u'(b) \end{aligned} \quad \left(\begin{matrix} |c_1| + |c_2| \neq 0 \\ |c_3| + |c_4| \neq 0. \end{matrix} \Leftrightarrow \left(\begin{matrix} c_1 \neq 0 & c_2 \neq 0 \\ c_3 \neq 0 & c_4 \neq 0 \end{matrix} \right) \right)$$

One can then write a general problem of the form

$$Lu = f(x)$$

$$B_1 u = 0$$

$$B_2 u = 0$$

∘ <u>Adjoint problems.</u>

$$(\mathcal{L}\phi, \psi) = \int_a^b (a_2\phi'' + a_1\phi' + a_0\phi)\bar{\psi} \quad \text{- Definition of } \mathcal{L},$$

$$= \int_a^b \phi\underbrace{\left((a_2\psi)'' - (a_1\psi)' + (a_0\psi)\right)}_{= L^*\psi} dx,$$

$$= (\phi, L^*\psi)$$

$$\Rightarrow (L\phi, \psi) = (\phi, L^*\psi).$$

$$L^*\psi = a_2\psi'' + (2a_2' - a_1)\psi' + (a_2'' - a_1' + a_0)\psi$$

if $a_1 = a_2' \Rightarrow \mathcal{L}^*\psi = \mathcal{L}\psi$

Then, it implies $\boxed{(\mathcal{L}\phi, \psi) = (\phi, \mathcal{L}\psi)}$ $\Rightarrow$ "Self adjoint"

E.g.) $\int u''v\,dx = \int u v''\,dx$

$$\underset{(\mathcal{L}u, v)}{\parallel} = \underset{(u, Lv)}{\parallel} \quad (\because a_1 = 0 = a_2' = 1' = 0)$$

$\begin{cases} \text{On } [a,b] \text{ define } (L\phi, \psi) - (\phi, L^*\psi) = J(\phi,\psi)\Big|_a^b \\ \text{where } J(\theta, \phi) = a_2(\phi'\bar{\psi} - \phi\bar{\psi}') + (a_1 - a_2')\phi\bar{\psi} \end{cases}$

if $J(\phi,\phi)\Big|_a^b = 0$, Then '$T$' is <u>self-adjoint</u>

$a_2(x) < 0$.

- A special case (particular choice).

$$p(x) = \exp\left(\int_a^x \frac{a_1(s)}{a_2(s)}\,ds\right) \qquad \left.\begin{aligned} w(x) &= -p(x)/a_2(x) \\ q(x) &= a_0(x)\cdot w(x) \end{aligned}\right\}$$

$p(x) > 0 \Rightarrow w(x) > 0 \in C([a,b])$.

Then, $\boxed{-(p\phi')' + q\phi = \lambda w(x)\phi \;,\; ((L_0\phi) = \lambda\phi)}$

↳ "Sturm – Liouville problem" $\qquad$ Eigenvalue problem. ✓

Let
$$\left.\begin{aligned} \mathcal{L}_1\phi &= (p\phi')' + q\phi \\ \mathcal{L}\phi &= \mathcal{L}_1\phi / w(x). \end{aligned}\right\}$$

Notice for ψ, $(L_1\phi, \psi) = (\phi, L_1\psi)$

$\because \int ((p\phi')' + q\phi)\psi\,dx \Rightarrow -\int \psi' p\phi'\,dx + \int q\phi\psi\,dx$

$\Rightarrow +\int (\psi' p)'\phi\,dx + \int q\phi\psi\,dx$

$\Rightarrow \int ((p\psi')' + q\psi)\phi\,dx = \underline{(\phi, L_1\psi)}$ #

$$\mathcal{L}\phi = \mathcal{L}_1\phi / w(x)$$

Claim: $L_0\phi = \lambda\phi$ iff $L\phi = \lambda\phi$

Define $(\phi, \psi) = \int_a^b \phi(x)\overline{\psi(x)}\,w(x)\,dx$

$$\|\phi\|_{L^2_w} = \left(\int_a^b |\phi(x)|^2 w(x)\,dx\right)^{1/2} = \sqrt{(\phi, \phi)}$$

↓ weighted L_2 space

$3h = \frac{180 \text{ min}}{4}$

$\cong 40$ min / problem.

$\partial^2/\partial\theta^2$ $\partial^2/\partial z^2$ (symmetric)

Problem Session

$\partial T/\partial t - \alpha \partial T/\partial x = 0$

$\partial T/\partial t - \alpha \Delta T = 0 \xrightarrow{1D} \partial T/\partial t - \alpha \left\{ \frac{1}{R} \frac{\partial}{\partial R}\left(R\, \partial T/\partial R\right)\right\} + \cancel{0} \; \cancel{0} \; 0$

$\Rightarrow \underline{\partial T/\partial t - \alpha \frac{1}{R} \partial/\partial R \left(R\, \partial T/\partial R\right) = 0}.$

① $-\partial T/\partial R = QR$, at $t = R$ — $\left(\ln (n'(i))^2\right)$ relative loss

② $\partial T/\partial R = h\,(T - T_\infty)$ at $b = R$

③ $T = T_\infty$ at $t = 0$

Substitute $\tau = \alpha t / R^2$, $\Theta = \frac{T - T_\infty}{QR}$ $R^* = N/R$ $N = hR$, $\tau = t/t_c$

$\Rightarrow \partial\Theta/\partial\tau = \frac{1}{R} \cdot \partial/\partial R \cdot \left(R \frac{\partial \Theta}{\partial R}\right)$ $\Theta = \tilde{\Theta} + \Theta_{ss}$.

$x \to E_b(f(x);\theta)) \to$ Loss $\to$ Back-propagate $\to |\ln(n'(i))|^2$ sort out batchsize

$\downarrow$

$x' \leftarrow$ Batch (new) = sorted indices

If it's set $n'(i) = 1$ well, sorting algorithm $\boxed{\text{will not}}$ sort out ①.

02/15/2024.

Lecture 11 :

we learned, $\begin{cases} \text{operator (differentiable)} \\ \text{adjoint} \\ \text{Sturm - Louiville problem.} \end{cases}$

$$a_2(x)\,\phi'' + a_1(x)\,\phi' + a_0(x)\,\phi = \phi\ \mathcal{L}_0[\phi] \qquad \forall x \in [a,b]$$

$$p(x) = \exp\left\{\int q_2(s)/a_1(s)\ ds\right\} \quad w(x) = -p(x)/a_2(x), \qquad q(x) = a_0(x)\,w(x)$$

Then, $\mathcal{L}_0\phi = \lambda\phi$ eigval problem is equivalent to,

$$\underbrace{(-p\phi')' + q\phi}_{L_1\phi \text{ is self-adjoint.}} = \lambda w \phi \qquad \underline{(\psi, L_1\phi) = (\phi, L_1\psi)}.$$

Recall a new weighted space,

$$L_w^2(a,b) = \left\{ \phi \ \middle|\ \|\phi\|_w^2 = \left(\int_a^b |\phi(x)|^2 w(x)\,dx\right)^{1/2} < \infty \right\}.$$

$$\text{Then, } \underline{(\phi,\psi)_w = \int_a^b \phi(x)\,\bar{\psi}(x)\ w(x)\ dx}$$

$$\boxed{(L\phi, \psi)_w = (\phi, L\psi)_w} \ \text{———}\ (*)$$

Boundaries,

$$B_1\phi = c_1\phi + c_2\phi'\Big|_a = 0 \qquad B_2\phi = c_3\phi(b) + c_4\phi'(b) = 0$$

$$J(\phi,\psi)\Big|_a^b = p(x)\left(\phi'\bar{\psi} - \phi\bar{\psi}'\right)\Big|_a^b = 0. \quad \text{If } \psi \text{ has same B.C.,} \ J(\phi,\psi)\Big|_a^b = 0.$$

$\therefore$ $\underline{T = \{B_1, B_2, L\} \text{ is self-adjoint space}}$

$\boxed{\text{Thm}}$

정리

a) $L_0 \phi = \lambda \phi$ is equivalent to $L\phi = \lambda\phi$.

b) $T = \{L, B_1, B_2\}$ is self-adjoint.

c) T has countable sequence of real, distinct eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \to \infty$

d) Corresponding eigenfunctions, $\{\phi_n\}$ may be chosen to form a basis of $L^2_w(a,b)$.

e) These eigenfunctions are orthogonal in the weighted space $L^2_w(a,b)$.

$$\frac{\int_a^b \phi_n \bar{\phi}_m \, w \, dx}{\int_a^b \phi_n \bar{\phi}_n \, w dx} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

f) Any $f \in L^2_w(a,b)$ can be written as $f = \sum_{n=1}^{\infty} c_n \phi_n$ & $c_n = (f, \phi_n)_w$

(Fourier)

* Normalization of orthogonal eigenfunctions.

$\{\phi_n\}_{n=1}^{\infty}$ are not orthonormal, they can be normalized using inner product.

e.g.) $\psi_n = \phi_n / (\phi_n, \phi_n)_w$

* Periodic B.C.s

$$\left.\begin{aligned} \phi(a) &= \phi(b) \\ \phi'(a) &= \phi'(b) \end{aligned}\right\} \Rightarrow \begin{aligned} \phi(a) - \phi(b) &= 0 \\ \phi'(a) - \phi'(b) &= 0 \end{aligned}$$

$\Rightarrow$ Not seperable $\boxed{\text{conditions for } a \,\&\, b \text{ are seperate}}$

$\Rightarrow \lambda = 0, \ \phi = 1$

$\boxed{\text{Note: } J(p, \phi) \text{ is still zero}}$

$\Downarrow$

Homogeneous.

still, (L, P_1, P_2) is still self-adjoint.

→ Additionally, eigenvalues are not simple.

↳ For each eigenvalue, there are two eigenfunction!

E.g.) Consider heat eq. $U_t = U_{xx}$ B.C. $u(0,t) = u(1,t) = 0 \quad \forall t > 0$

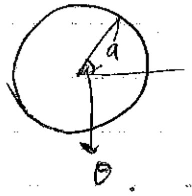
$u(x,0) = f(x) \qquad 0 < x < 1$

$$u_k = c_k e^{-k^2\pi^2 t} \sin(k\pi x)$$

$$u = \sum_{k=1}^{\infty} A_k \exp(-k^2\pi^2 t) \sin(k\pi x) \quad \text{(generalized Fourier series)}$$

X_A — $\boxed{1+2}$

Eg. Heated cylindrical rod. $x = a\cos\theta$, $y = a\sin\theta$.



$$\partial T/\partial t = \nabla^2 T$$

Using chain-rule,

$$\partial T/\partial t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\, \partial T/\partial r\right) \qquad (\theta - \text{symmetric})$$

$$\left.\begin{array}{l} T(r,0) = f(r) \\ T(0,t) \text{ is finite} \rightarrow \text{No blow up.} \\ T(a,t) = 0 \end{array}\right\}$$

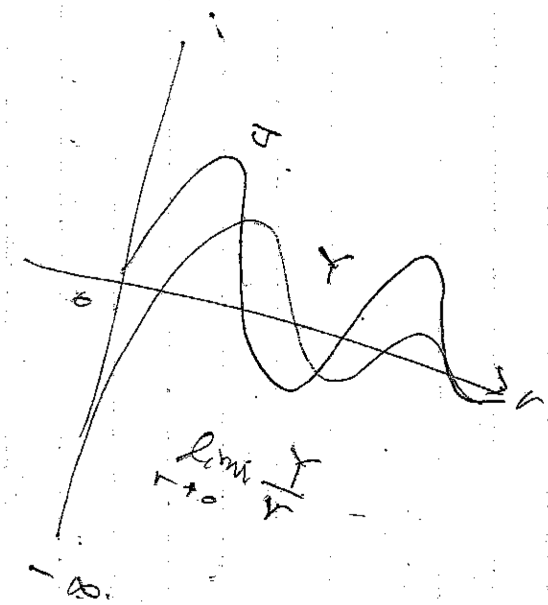
SOV $\Rightarrow \underline{T(r,t) = R(r)\tilde{T}(t)}$

$$\Rightarrow \frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \frac{T'}{T} = \alpha \qquad \alpha = -\lambda^2 \rightarrow R(a) = 0$$

$$\Rightarrow \boxed{r^2 R'' + rR' + r^2\lambda^2 R = 0}$$

where Bessels' equation of order ν is. $x^2\phi'' + x\phi + (x^2 - \nu^2)\phi = 0$.

where $\phi = c_1 \underbrace{J_\nu(x)}_{1^{st}\text{ kind}} + c_2 \underbrace{Y_\nu(x)}_{2^{nd}\text{ kind}}$



$$\lim_{r \to 0} \frac{Y}{r}$$

$10^{-6} \sim 10^{-8}$ 100

TA session 02/16/2024

- Sturm Loiville problem

$$\mathcal{L} y = -(p(x)y')' + q(x)y = \lambda r(x) y$$

$$\mathcal{L}y / r(x) = \lambda y \rightarrow \boxed{\mathcal{L}_1 y = \lambda y}$$

B.C. ① $B_1 u = c_1 y(a) + c_2 (y'(a)) = 0$

② $B_2 u = c_3 y(b) + c_4 y'(b) = 0$

- All eigenvalues are real, distinct and can be ordered.
  $\lambda_1 < \lambda_2 < \cdots < \infty$
- Eigenvectors of distinct eigenvalues are ~~diagonal~~ orthogonal to each other.
  $\langle \phi_m, \phi_n \rangle = \int_a^b r(x)\, \phi_m \phi_n \, dx = N_m \delta_{m,n}$ $\quad (\mathcal{L}_1(\phi_m) = \lambda_m \phi_m)$ $\quad N_m = \langle \phi_m, \phi_m \rangle$
- Any function in L^2 and satisfy B.C,
  $f(x) = \sum_{m=1}^{\infty} c_m \phi_m(x)$ where $\left(c_m = \dfrac{\langle f, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle} \right)$

Bessel's equation.

$$t^2 y'' + t y' + (t^2 - \nu^2) y = 0$$

$y = c_1 J_\nu(t) + c_2 Y_\nu(t)$

J_ν: Bessel func. 1st kind order ν

Y_ν: Bessel func. 2nd kind order ν

$\lambda_m^2 t^2 - \nu^2$ → $J_\nu(\lambda_m t)$, $Y_\nu(\lambda_m t)$

Modified Bessel equation

$$t^2 y'' + t y' - (t^2 + \nu^2) y = 0$$

$y = c_1 I_\nu(t) + c_2 K_\nu(t)$

I_ν: $= + i^{-\nu} J_\nu(ix)$ → Not oscillatory in nature.

K_ν: $\frac{\pi}{2} \left\{ \dfrac{J_{-\nu}(x) - J_\nu(x)}{\sin \pi\nu} \right\}$ → Singular at $t=0$

Note : $-(py')' + q(x)y = \lambda r(x) y$ ~~becomes~~, to express this,

$$x^2 y'' + x y' + (\lambda x^2 - \nu^2) y = 0. \quad (t = \sqrt{\lambda} x)$$

↓

$$\frac{d}{dx}(x y') + \frac{\nu^2}{x} y = \lambda x y \Rightarrow \begin{array}{l} p(x) = x \\ q(x) = \nu^2/x \\ r(x) = x \end{array}$$

domain $[0,1]$.

$r(x)$: weight function.

For S.L problem

B.C. ① $y(1) = 0$

② y is well-behaved at 0.

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

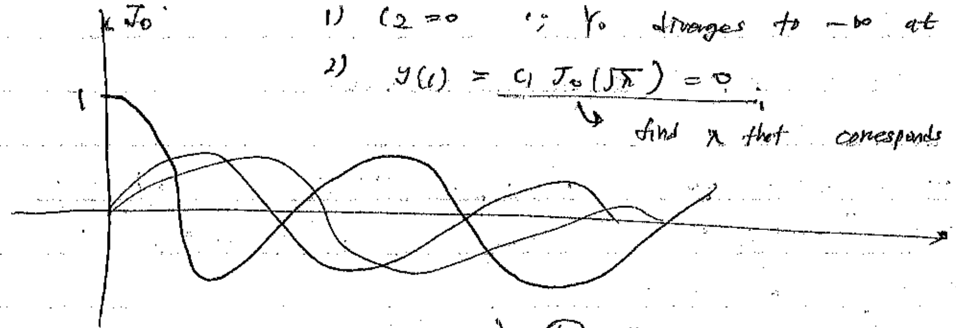
$$\begin{array}{cc} \uparrow & \uparrow \\ \sqrt{\lambda}x & \sqrt{\lambda}x \end{array}$$

For $\nu = 0$, $y(x) = c_1 J_0(\sqrt{\lambda}\, x) + c_2 Y_0(\sqrt{\lambda}\, x)$

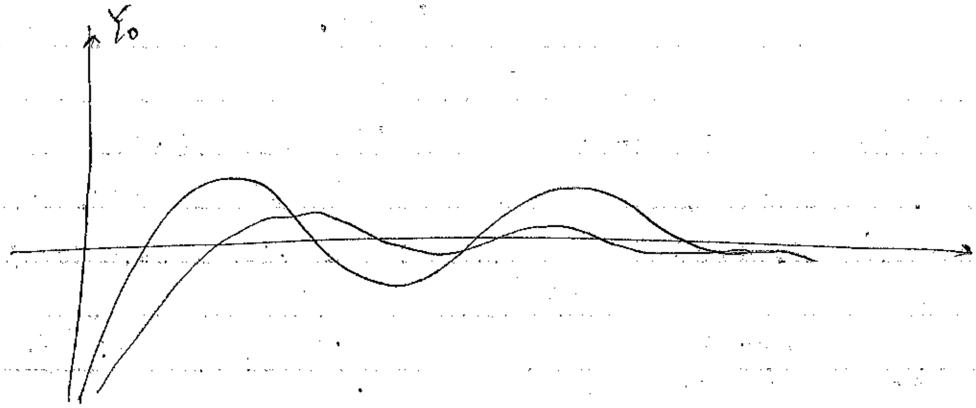
1) $c_2 = 0$ ∵ Y_0 diverges to $-\infty$ at $x=0$.

2) $\underline{y(1) = c_1 J_0(\sqrt{\lambda}) = 0}$

↳ find λ that corresponds to node where $J = 0$



$\lambda = ①$ λ_1

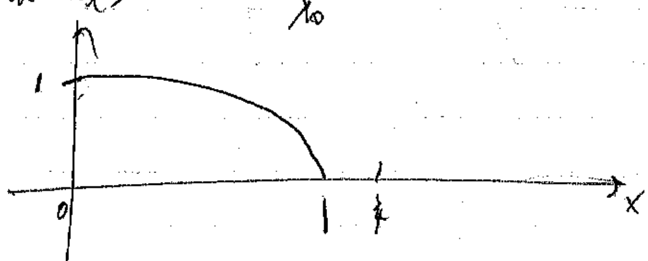


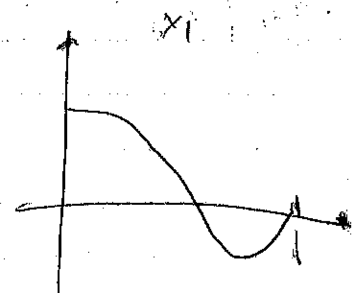
$$\int_0^1 J_0(\lambda_i x)\, J_0(\lambda_j x)\, dx = 0 \quad m \neq n$$

x

weight function !

<Final sol>





Lecture 02/20/2024

- ODEs of S.L. problem.

$r^2 R'' + rR' + \lambda^2 r^2 R = 0$

$\Rightarrow \ -(rR')' = \lambda^2 r R$

$\phi(r) = r$

$\boxed{w(r) = r} \longrightarrow (f, g)_w = (f, g)_r$ (needed for orthogonalization)

comparing to. $\underline{-(p\phi')' + q\phi = \lambda w \phi}$

- Bessel $-p$

$x^2\phi'' + x\phi' + (x^2 - \nu^2)\phi = 0 \rightarrow \phi(x) = \underline{c_1 J_\nu(x)} + \underline{c_2 Y_\nu(x)}$

$\lambda^2 r^2 \phi'' + \lambda r \phi' - \lambda^2 r^2 \phi = 0$

$\hookrightarrow \underline{\phi = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)}$

$\downarrow$

Note: We can only handle one non-homogenity.

B.C. $R(1) = 0 \rightarrow J_0(\lambda) = 0 \Rightarrow \lambda = z^n \rightarrow$ zeros of bessel.

Eig. functions are $J_0(z_n r)\Big|_{n=1}^{\infty}$

$N_n = \int_\Omega J_0(z_n r)\, J_0(z_n r)\, \underbrace{r}_{\text{weight function.}}\, dr = N_n\, \delta_{n,m}.$

$T(r,t) = \sum_{n=1}^{\infty} c_n J_0(z_n r)\, e^{-z_n^2 t}$

$T(r, t=0) = \sum_{n=1}^{\infty} c_n J_0(z_n r) = f(x).$

$\Rightarrow c_n = \dfrac{(f(r),\ J_0(z_n r))_w}{(J_0(z_n r)\, J_0(z_n r))_w}$

Schrödinger's equation. 02/22/2024.

$$\psi(r,\theta,\phi) = R(r)\,\Theta(\theta,\phi)$$

$\downarrow$

S.O.V

$\downarrow$

$\mathcal{L}_1\{R(r)\} = \mathcal{L}_2\{\Theta(\theta,\phi)\} = \text{constant} \rightarrow$ solve for $R(r)$. !

$\downarrow$

S.O.V. to $\Theta(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$

$\downarrow$

solve for $\Theta(\theta)$ and $\Phi(\phi)$ 2 3

Algorithm.

Tip ① Using $\Phi(\phi) = \Phi(\phi + 2\pi)$

$$\exp(jm\phi) = \exp(jm\cdot(\phi+2\pi)) \Rightarrow$$ 'm' must be integers.

Tip ② → You'll obtain.

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \{\ell(\ell+1)\sin^2\theta - m^2\}\,\Theta = 0$$

Associated Legendre equation

→ $P_\ell^m(x)$ and $Q_\ell^m(x)$ will be used

$$Y_\ell^m(\theta,\phi) = N_{\ell m}^{-1}\, e^{jm\phi}\, P_\ell^m(\cos\theta)$$

$$N_{\ell m}^{-1} = \varepsilon\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}$$

$$\varepsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases}$$

Let $u = rR(r) \rightarrow R = u/r$, so the original eq. can be expressed into,

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r) + \frac{\hbar^2}{2m}\cdot\frac{\ell(\ell+1)}{r^2}\right)\cdot u = Eu$$

mass mass

$$-1 \le x \le 1$$
$$0 < \theta < \pi$$

TA session. 02/22/2024.

① $\partial T/\partial t = \alpha \nabla^2 T = \alpha\left(\frac{1}{R^2}\frac{\partial}{\partial R}(r^2\, \partial T/\partial r) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta \frac{\partial T}{\partial \theta}\right)$

$\rightarrow \boxed{\text{sub}}$ $\Theta = T/\Delta T$ $\quad \xi = r/R,\ \tau = \alpha t/R^2$

I.C. $T(t=0) = \Delta T$
B.C. $T(r=R) = 0$
$T(r=0) =$ finite.

② $\Theta = T(\tau)\cdot H(\xi,\theta)$ $\quad \partial\Theta/\partial\tau = \nabla^2_{(\xi,\theta)}\Theta$ $\quad$ SOV1

$HT' = T\nabla^2_{\xi,\theta} H \Rightarrow T'/T = \frac{1}{H}\nabla^2_{(\xi,\theta)} H = -\lambda^2$ ($\because$ T'/T term) not diverge.

$\Rightarrow 1/\xi^2\, \partial/\partial\xi\left(\xi^2 \frac{\partial H}{\partial \xi}\right) + \frac{1}{\xi^2 \sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial H}{\partial\theta}\right) = -\lambda^2 H.$

③ $H = R(\xi)\Psi(\theta)$ $\quad$ SOV2

$-\frac{1}{R}\left\{\frac{\partial}{\partial \xi}\xi^2\frac{\partial R}{\partial \xi} + \lambda^2 R\xi^2\right\} = \frac{1}{\Psi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) = -\beta^2$ (why $-\beta^2$!)

<θ - space> — β

S.L. problem using Legendre, eq..

$\because$ $P_m^l(x)$ are finite at the end points.

$0<\theta<\pi$, finite (should be).

<ξ space> — λ $\quad \frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial R}{\partial\xi}\right) - \frac{m(m+1)}{\xi^2}R = -\lambda^2 R.$ $\quad \because$ $R(\xi=1)=0.$
$R(\xi=0) \rightarrow$ finite $<\infty$

$\Rightarrow$ β spherical

$R = A\, j_m(\lambda\xi) + B\, \cancel{y_m(\xi)} \rightarrow 0$ ($<\infty$ condition)

$\Rightarrow R(\xi) = j_m(\lambda_m \xi)$

$\therefore H = R(\xi)\,\Theta(\theta)$

$\Theta(\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\exp(-\lambda_{mn}\tau)\cdot j_m(\lambda_{mn}\xi)P_m(\cos\theta)$

$$\int_{\theta=0}^{\pi} \sin\theta\, f(\xi,\theta) P_S(\cos\theta)\, d\theta = \sum_{m=1}^{\infty} A_{sm}\, j_s(x_{sm}) \frac{2}{2S+1}$$

Now, integrate respect to ξ

$$\Rightarrow \int_0^1 \xi^2 G_s(\lambda_{sn}\xi) \int_{\theta=0}^{\pi} \sin\theta\, f(\xi,\theta) P_s(\sin)\, d\omega =$$

Example problem HW5 #3.

$\Delta u = 4$, $\quad u(x,y) = 1 \quad r=1 \quad y>0$

$u(x,y) = 0 \quad r=1 \quad y<0.$

$\underline{u = r^2 + u_H}$

$\Rightarrow \quad \Delta u_H + 4 = 4 \quad \Rightarrow \quad u_H = 0 \quad$ where $\left.\begin{array}{ll} u_H = 0 & y>0 \\ u_H = -1 & y<0. \end{array}\right\}$ new problem.

$$\left(\begin{array}{lll} R\Theta = 0 & r=1 & 0<\theta<\pi \\ R\Theta = -1 & r=1 & \pi<\theta<2\pi. \end{array}\right) \text{B.C.s.}$$

$$\Rightarrow \left\{\begin{array}{ll} \Theta(\theta) = 0 & 0<\theta<\pi \\ \Theta(\theta) = \underbrace{-\,1/R(1)}_{\text{constant.}} & \pi<\theta<2\pi. \end{array}\right.$$

Schrödinger eq. continued.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\left(-\frac{2mr^2}{\hbar}\right)\left[V(r)-E\right]R = \underline{l(l+1)R}$$

radial term.

$$V(r)=\frac{-e^2}{4\pi\varepsilon_0}\frac{1}{r}$$

Substitution : $u = rR(r)$, $R = u/r$

$$\underline{dR/dr = \{r\,\partial R/\partial r + (-u)\}/r^2}$$

Remap.

$$\rightarrow \; +r\,u_{rr} + Rmr/\hbar^2\; V(r) + l(l+1)R = \frac{2mr}{\hbar^2}Eu$$

multiply by $\hbar^2/2mr$

$$\Rightarrow -\frac{\hbar^2}{2m}u_{rr} + V(r) + l(l+1)\,R\left(\frac{r}{r}\right)\hbar^2/2mr = Eu$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + \left\{V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right\}u = Eu.$$

let $\underline{\rho = kr}$ & $\underline{\rho_0 = me^2/2\pi\varepsilon_0\hbar^2 k}$

$$\Rightarrow \partial^2 u/\partial\rho^2 = \left[1-\frac{\rho_0}{\rho}+\frac{l(l+1)}{\rho^2}\right]u$$

As $(kr)\to\infty$,
$\underline{\partial^2 u/\partial\rho^2 = u}$
$\Rightarrow u = Ae^{-\rho}+Be^{\rho}$.

$e^{\rho}\to\infty$ as $\rho\to\infty$
$u(\rho)\sim Ae^{-\rho}$
(large ρ)

$\rho\to 0$. $\frac{l(l+1)}{\rho^2}$ dominates.

$$\frac{\partial^2 u}{\partial\rho^2} = \frac{l(l+1)u}{\rho^2}$$

$$\Rightarrow \rho\frac{\partial^2 v}{\partial\rho^2} + 2(l+1-\rho)\frac{\partial v}{\partial\rho} + (\rho_0 - 2(l+1))v = 0.$$

$\nu = 2l+1$, $x = 2\rho$.

We obtain, $x\phi'' + (\nu+1-x)\phi' + \lambda\phi = 0$.

$\downarrow$ Associated Laguerre f.

$$\underline{L^{p}_{q-p}(x) = (-1)^p\left(\frac{d}{dx}\right)^p L_q(x)}$$

$$L_q(x) = e^{x}\left(\frac{d}{dx}\right)^{q}\left(e^{-x}x^{q}\right)$$

where $a_{j+1} = \left\{\frac{2(j+l-1)-\rho_0}{(j+1)(j+2l-2)}\right\} a_j \qquad \left(\left(a(j)_{max}\right)=0\right)\Big|_{2(j_{max}+l+1)-\rho_0=0}$

let $n = j_{max}+l+1$

$\underline{2n-\rho_0=0} \Rightarrow n$

$\downarrow$

(principal quantum #).

$$\boxed{E_n = -\left\{\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right\}\frac{1}{n^2} \qquad \forall n=1,2,\ldots}$$

$\hookrightarrow$ Bohr formula

$$\Rightarrow \psi_{n,l,m}(r,\theta,\phi) = R_{nl}(r)\,Y_l^m(\theta,\phi)$$

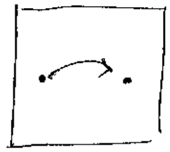
$$R_{nl}(r) = 1/r\;\rho^{l+1}e^{-\rho}\,v(\rho)$$

$$\underline{\& \; v(\rho) = L^{2l+1}_{n-l-1}(2\rho)}.$$

$$\psi_{nlm} = \left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n((n+l)!)^3}\, e^{-r/na}\left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right)Y_l^m(\theta,\phi).$$

$$dX_t = -\nabla E \, X_t + \varepsilon . N(0, \Delta t).$$

$$\Delta x = -\nabla E \, \Delta t + \varepsilon N(0, \Delta t)$$



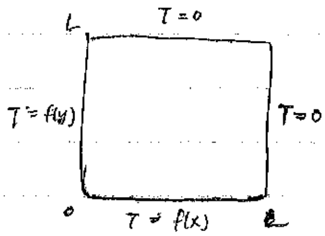
- Inhomogeneties.

Linear PDE for S.O.V.

Separable $\left\langle \begin{array}{l} \text{SOV ansatz} \\ \text{gives ODEs.} \end{array} \right|$ ⇒ ~~danates~~ 1차 미분임. . .

All but 1 condition have to be homogeneous.

E.g. Simple inhomogeneity.



$$T(x, y) = T_1 + T_2$$

$$\begin{array}{ll} T_1(x, 0) = f(x) & T_2(x, 0) = 0 \\ T_1(x, L) = 0 & T_2(x, L) = 0 \\ T_1(0, y) = 0 & T_2(0, y) = g(y) \\ T_1(L, y) = 0 & T_2(L, y) = 0 \end{array}$$

S.O.V & Transformations. 02/29/2024.

- Inhomogeneities $\Rightarrow$ Idea of Superposition (Linear).

$T_{xx} + T_{yy} = 0$ $\qquad T(x,0) = f(x) \quad T(x,L) = 0$

$T(0,y) = g(y) \quad T(L,y) = 0$

$$\Rightarrow T(x,y) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi y}{L}\right) \sinh\left(\frac{n\pi(L-x)}{L}\right) \longrightarrow \text{How to calculate } A_n?$$

by linearity, $T(x,y) = T_1(x,y) + T_2(x,y)$.

T_2 will satisfy $\quad T_2(x,0) = \boxed{f(x)} \quad T_2(x,L) = 0$

$T_2(0,y) = 0 \quad T_2(L,y) = 0$

T_1 will satisfy $\quad T_1(x,0) = 0 \quad T_1(x,L) = 0$

$T_1(0,y) = \boxed{g(y)} \quad T_1(L,y) = 0.$

One inhomogeneity at a time! (solve for

2 inhomogeneity $\rightarrow$ divided into 1 homogeneity $\rightarrow$ Add again

- General inhomogeneities.

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t} - f(r,t) \qquad f(r,t) = \sum_{n=1}^{\infty} c_n \phi_n \qquad \begin{pmatrix} u(r=0,t) < \infty \\ u(r,t=0) = 0. \end{pmatrix}$$

$T(t) = a e^{-\lambda^2 t}$, $R(r) = a\, J_0(z_n r)$

$u = \sum R(r)\, T(t)$ (Sol of Homo)

$u = u^h + u^p \rightarrow$ (sol to inhomo with zero B.C.)

$\downarrow$ (sol to homo)

$$u^p(r,t) = \sum_{n=1}^{\infty} A_n(t)\, J_0(z_n r) \rightarrow \text{substitute back into PDE}$$

$$\sum_n A_n(t)\left(-J_0(z_n r)\right)'' = \sum_n \frac{dA_n(t)}{dt} J_0(z_n r).$$

$\qquad ||$ $\qquad\qquad\qquad\qquad\qquad ||$

$\sum_n A_n(t)\, z_n^2\, J_0(z_n r)$ $\qquad \sum_n \frac{dA_n(t)}{dt} J_0(z_n r) - c_n(t)\, J_0(z_n r)$

($\because$ property of Bessel func)

since orthogonal, $\quad -A_n(t)\, z_n^2 = -c_n(t) + dA_n(t)/dt$

Therefore, $dA_n(t)/dt = C_n(t) - A_n(t)\lambda_n^2$

$$\begin{cases} A_n(t) = \int_0^t e^{\lambda_n^2 t'} C_n(t')\,dt' \\ C_n(t) = \dfrac{\langle f(x,t), \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \end{cases}$$

— End of S.O.V —

$$\frac{G(x-\xi, t-t')}{G(x,t)}$$

$x \to x-\xi$

$t \to t-t'$

Fourier transform 02/29/2024.

$$\mathcal{F}(f(x)) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} f(\xi)\, e^{jk\xi}\, d\xi = \hat{f}(k)$$

$$f(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{-jkx}\, dx$$

$$\mathcal{F}\{f(x)\} = f(k) = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, e^{ajk\xi}\, d\xi$$

$$f(x) = \frac{1}{\cancel{\gamma}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{-ajkx} dk$$

Choose $\gamma = \sqrt{2\pi}$, $a = +1$ → makes sense (when $f, \hat{f}$ are integrable)

* Properties

1) $\mathcal{F}\{f'\} = -jk\,\mathcal{F}\{f\}$

$\mathcal{F}\{f''\} = -k^2\,\mathcal{F}\{f\}$

2) Convolution.

$\mathcal{F}\{f\} = \hat{f}(k)$

$\mathcal{F}\{g\} = \hat{g}(k)$

$\mathcal{F}\{f * g\} = \hat{f}(k)\;\hat{g}(k)$

convolution.

$$f * g = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi)\, f(x-\xi)\, d\xi$$

* Fundamental sol.

$$u_t - u_{xx} = \phi(x,t) \qquad -\infty < x < \infty$$

Formulate as, $\boxed{u_t - u_{xx} = \delta(x-\xi)\cdot\delta(t-t')}$

solution is $F(x-\xi,\, t-t')$

Then, we know solution is

$$u(x,t) = \int_{-\infty}^{\infty} \int_0^t F(x-\xi,\, t-t')\cdot \phi(\xi, t')\, dt'\, d\xi'$$

ξ''

$$u_t - u_{xx} = \delta(x-\xi)\,\delta(t-t')$$

$$\hat{u}_t + k^2 \hat{u} = \frac{\delta(t-t')}{\sqrt{2\pi}} \int \underbrace{\cancel{\delta(x)}\, e^{jkx}\, dx}_{\delta(x-\xi)} \qquad = e^{jk\xi}$$

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-k^2 t} e^{-jkx}\, dk$$

Solve for $\xi = 0,\ t' = 0$ ⇒ $\hat{u}_t + k^2 u = \delta(t)/\sqrt{2\pi}$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-k^2 t} e^{-ikx} \, dk$$

if $m = \xi + i \frac{x}{2t^{1/2}}$ $\Rightarrow$ Then e^{-m^2} is analytic.

$$\Rightarrow G(x,t) = \sqrt{\pi} \, e^{x/2t^{-1/2}}$$

- Calculate rates.

1) states $= (-1, 1, 0)$

2) rate $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ p_1 & p_2 & p_3 & p_4 \end{bmatrix}$

3) $k = [\overset{0}{p_1} \quad \overset{0}{p_2} \quad \overset{0}{p_3} \quad \overset{0}{p_4}]$

4) prates $= (p_1', p_2', p_3', p_4')$.

5) if. <u>success</u>

Initial : states $= (-1, 1, 0)$ rates $= (p_1 \quad p_2 \quad p_3 \quad p_4)$

Fourier Transform - example.

- $\widehat{\partial\phi/\partial x} = jk\phi$
- $\widehat{\partial^2 u/\partial x^2} = \widehat{\frac{\partial}{\partial x}\frac{\partial u}{\partial x}} = jk\,\widehat{\partial u/\partial x} = jk\,jk\,(\hat{u}) = -k^2\,\hat{u}\,(k,t).$
- $\widehat{\partial^2 u/\partial t^2} = \partial^2\hat{u}/\partial t^2$

$$\begin{cases} \partial^2 u/\partial t^2 + (-c^2)\,\partial^2 u/\partial x^2 = 0 \\ u(x,0) = f(x) \\ \partial u/\partial t\,(x,0) = 0 \end{cases} \xrightarrow{\mathcal{F}} \begin{array}{l} \partial^2\hat{u}/\partial t^2 + c^2 k^2 u^2(k,t) = 0 \\ \hat{u}(k,t) = \hat{A}(k)\exp(-jkt) \\ \hat{u}(k,0) = \hat{f}(k) \\ \partial\hat{u}(k,0)/\partial t = 0. \end{array}$$

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\,.\,e^{-k^2 t - jkx}\,dk$$

if $\hat{f}(k)\;e$

$f(x)\;*$

$$\phi(x,t) = f(x) * \mathcal{F}^{-1}\{e^{-k^2 t}\}$$

$$e^{-x^2} * f(x) = \qquad f(\tau)\,e^{-(x-\tau)^2}$$

1. $\underline{\partial T/\partial t = \alpha \nabla^2 T} = \underline{\alpha\, \partial^2 T/\partial x^2}$.

(a)

	$\tilde{T}$	x	t	β	α
degrees	1	0	0	1	0
length	0	1	0	-1	2
time	0	0	1	-1	-1

$m = 3,\ n = 5$

$n - m = \underline{2\ \pi \text{ groups}}$.

$\tilde{T} = T - T_0$ (B.C. is still).

$$\left.\begin{array}{l} \tilde{T}(x, t=0) = 0 \\ \tilde{T}(x \to \infty, t) \to T_0 \\ -T_x(x=0, t) = \beta t \end{array}\right\} \text{ satisfies for } \tilde{T}$$

$\underline{\Pi_1 = \text{fnc}(\Pi_2)}$

(b) $\tilde{T} = T - T_0 = B\, t^m F(\eta) \qquad \eta = x/(A t^n)$

LHS : $\partial \tilde{T}/\partial t = m B t^{m-1} F(\eta) + B t^m F'(\eta).(x/A).\{-n t^{-n-1}\}$

RHS : $\alpha\, \partial^2 T/\partial x^2 = \alpha\, \partial/\partial x \{ B t^m F'(\eta). \frac{1}{A t^n}\} = \alpha. B t^m. F''(\eta)\left(\frac{1}{A t^n}\right)^2$

$\Rightarrow$ t order should match $\rightarrow$ $m - n - 1 = m - 2n \Rightarrow n = \frac{1}{2}$

B.C. : $-T_x(x=0, t) = \left. -B t^m F'(\eta). \frac{1}{A t^n} \right|_{x=0} = \beta t^1$

$\Rightarrow m - n = 1 \Rightarrow \boxed{m = \frac{3}{2}}$

PDE becomes, $\frac{3}{2} B t^{1/2} F(\eta) + B t^{3/2} F'(\eta) \frac{x}{A}\left(-\frac{1}{2} t^{-3/2}\right) = \alpha B t^m F''(\eta)\left(\frac{1}{A t^n}\right)^2$

$2F'' + \eta F' - 3F = 0,\ F(0) = 1,\ F(\infty) = 0.$

$\boxed{A = \sqrt{\alpha}}$ $\boxed{B = -\beta\sqrt{\alpha}}$

(c) Superposition.

2. (a)

	c	x	t
deg.			
time			
length			

$n = 1/2 \quad m = 0$

$0 = F''F + (F')^2 + 2hF \quad F(0) = 1, F$

$$\begin{array}{c|cccc} & \overbrace{\tilde{H} \quad x \quad t \quad \alpha}^{n} \\ \hline m \begin{cases} \text{Length} \\ \text{time} \\ \text{depth.} \end{cases} & \end{array}$$

$n - m = 1 <$ # var.

3. $\partial H/\partial t = -\frac{\gamma}{3\mu} \frac{\partial}{\partial x} \{ H^3 \partial^3 H/\partial x^3 \}$

(a) $H(x,t) = \underline{\beta t^{m}} F(\eta)$ where $\eta = \frac{x}{At^n}$

LHS : $\partial H/\partial t = m \beta t^{m-1} F(\eta) + \beta t^{m} F'(\eta) . (-n) \frac{x}{At^{n+1}}$

$$= \underline{\beta t^{m-1} \{ mF - n\eta F' \}}$$

RHS : $-\frac{\gamma}{3\mu} \frac{\partial}{\partial x} \{ H^3 \partial^3 H/\partial x^3 \} = -\frac{\gamma}{3\mu} \{ 3H^2 \underline{\partial H/\partial x} \underline{\partial^3 H/\partial x^3} + H^3 . \underline{\partial^4 H/\partial x^4} \}$

$$\partial H/\partial x = \beta t^{m} . F'(\eta) \frac{1}{At^n} = \frac{\beta}{A} t^{m-n} F'(\eta) .$$

$$\partial^2 H/\partial x^2 = \beta/A^2 \; t^{m-2n} F''(\eta)$$

$$\partial^3 H/\partial x^3 = \beta/A^3 \; t^{m-3n} F'''(\eta)$$

$$\partial^4 H/\partial x^4 = \beta/A^4 \; t^{m-4n} F''''(\eta)$$

RHS : $-\frac{\gamma}{3\mu} \{ 3 . \beta^2 t^{2m} (F(\eta))^2 \; \frac{\beta}{A} t^{m-n} F'(\eta) . \frac{\beta}{A^3} t^{m-3n} F'''(\eta)$

$$+ \beta^3 t^{3m} F^3 \frac{\beta}{A^4} . t^{m-4n} F''''(\eta) \}$$

$$= -\frac{\gamma}{3\mu} \{ 3 \underline{\beta^4/A^4} . \{F(\eta)\}^2 F'(\eta) F'''(\eta) . \underline{t^{4m-4n}}$$

$$+ \underline{\beta^4/A^4} F^3 F''''(\eta) . \underline{t^{4m-4n}} \}$$

$\therefore$ $m - 1 = 4m - 4n . \Rightarrow \underline{3m = 4n - 1}$ ($\because$ RHS = LHS).

and

$m = 1/7 \quad n = -1/7 \quad$ ($\because$ $m + n = 0$).

from (12) – integral condition.

$C_p / C_v = \gamma$

$P V^{\gamma} = C_0$

4. $k \propto \sqrt{T}$ $\quad C_p \propto \gamma T^2$

$\frac{\partial T}{\partial x}(x=0,t) = 0$

$T(\infty, t) = 0.$ $\qquad \partial/\partial x \{ \sqrt{T}\, \partial T/\partial x \} = \gamma T^2\, \partial T/\partial t$

$T(x,0) = 0$

$\gamma \int_0^{\infty} T^3\, dx = \beta.$

	T	x	t	γ	β
{L}	0	1	0	−2	
{T}	0	0	1	~~[illegible]~~ 1	
{K}	1	0	0	−3/2	

$n - m = 5 - 3 = 2$

$\to \pi_1, \pi_2 = f^{n}(\pi_1)$

exists!

sim. sol

let, $\pi_1 = x^a t^b \gamma^c \beta^d = \eta$

$\pi_2 = T^a t^b \gamma^c \beta^d = \theta \to$ $\quad a - 2c - d = 0 \quad \to a = 1$

$b + c + d = 0 \qquad b = -2/3$

$3/2(d - c) = 0 \qquad c = d = 1/3$

set same (default).

$1 - \frac{3}{2}c + \frac{3}{2}d = 0$

$-2c - d = 0$

$\underline{d = -2c}.$

$2c = -d$

$1 - \frac{3}{2}c + \frac{3}{2}(-2c) = 0$

$1 - \frac{3}{2}c - 3c = 0$

$c = \frac{2}{9}. \quad d = -\frac{4}{9}$

$$\nabla V \cdot \nabla q - \beta^{-1} \nabla^2 q = 0.$$

$$q = \exp\left(-\frac{\beta}{2} \cdot E_b \right)$$

$$\cancel{\nabla q = -\nabla E_b \exp\left(-\frac{E_b}{2k_BT}\right)}.$$

$$\nabla q = -\frac{\beta}{2} \cdot \nabla E_b \exp\left(-\frac{\beta}{2} E_b\right) = \left(-\frac{\beta}{2} \nabla E_b\right) \boxed{\exp}$$

$$\nabla^2 q = -\frac{\beta}{2} \nabla^2 E_b \exp\left(-\frac{\beta}{2} E_b\right) + \left(\frac{\beta}{2} \nabla E_b\right)^2 \exp\left(-\frac{\beta}{2} E_b\right) = 0$$

$$= \left\{ -\frac{\beta}{2} \nabla^2 E_b + \left(\frac{\beta}{2} \nabla E_b\right)^2 \right\} \boxed{\exp}$$

$$\nabla V \left(-\frac{\beta}{2} \nabla E_b\right) \cancel{\exp} - \beta^{-1} \left\{ -\frac{\beta}{2} \nabla^2 E_b + \left(\frac{\beta}{2} \nabla E_b\right)^2 \right\} \cancel{\exp} = 0.$$

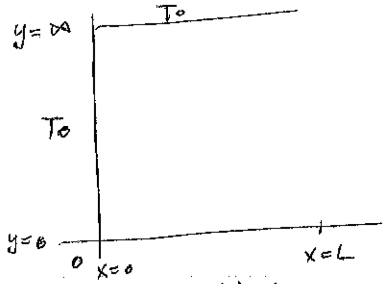
$$\Rightarrow -\frac{\beta}{2} \nabla V \nabla E_b - \beta^{-1} \left\{ -\frac{\beta}{2} \nabla^2 E_b + \frac{\beta^2}{4} (\nabla E_b)^2 \right\} = 0$$

$$\Rightarrow -\frac{\beta}{2} \nabla V \nabla E_b - \frac{\beta}{4} (\nabla E_b)^2 + \frac{1}{2} \nabla^2 E_b = 0$$

$$\Rightarrow \nabla^2 E_b = \beta \nabla V \nabla E_b + \frac{\beta}{2} (\nabla E_b)^2$$

B.C: $\exp\left(-\frac{\beta}{2} E_b\right) = 0 \quad (x \in \partial A)$

$\exp\left(-\frac{\beta}{2} E_b\right) = 1 \quad (x \in \partial B)$



5. $\beta y^{\gamma}\, \partial T/\partial x = \alpha\, \partial^2 T/\partial y^2$

T

a) Using algebraic similarity,

b) $\tilde{T} = T - T_0 \quad \xi = x/L \quad , \eta = \beta y.$

$$\partial T/\partial x = \underline{\partial \tilde{T}/\partial T} \cdot \left(\partial T/\partial \tilde{T}\right) \cdot \underline{\partial T/\partial x} \quad \underline{\partial x/\partial \xi} \left(\partial \xi/\partial x\right)$$

$$= \partial \tilde{T}/\partial \xi \cdot 1 \cdot \frac{1}{L}$$

$$\tilde{T} = T - T_0 = B x^m F\left(\frac{y}{A x^n}\right)$$

$m = 1/(2+\gamma) \quad , n = 1/(2+\gamma)$

$A = k,\ B = -q$ (you chose),

$F'(0) = 1,\ F(\infty) = 0$

$$\frac{\alpha}{k^2} F'' - \beta \eta^{\gamma} k^{\gamma} \left(\frac{F}{2+\gamma} - \frac{\eta F'}{2+\gamma}\right) = 0.$$

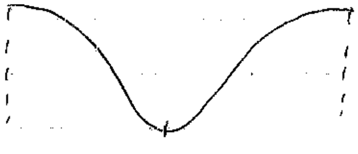
$$\Rightarrow (2+\gamma) F'' = \eta^{\gamma} (F - \gamma F')$$

$$T - T_0 = B x^m F\left(\frac{y}{A x^n}\right)$$

6. $m = -1 \qquad C = B t^m F\left(\gamma / A x^n\right)$

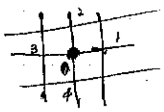
$n = 1/2$

$$\int A e^{-r^2} = 1$$



$k(i \to 0) = 0$

$q(0) = 0 = k(0 \to 1)$



$q(0) = k(0 \to 1) \cdot q(1) + \dots + k(0 \to 4)\, q(4)$ (contradiction).

$q(N) = 1$

10

11

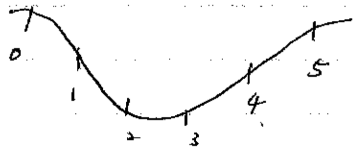
8979323846

e^{-x^2}

$1/\sqrt{2} = \sigma$

$2\sigma^2 = 1$

0.8409



$q(0) = 0$

F S are cemetery states.

$k(x \to y) \Rightarrow k(x \to y) \cdot (1 - f(x) - s(x))$ (continue RW).

$f(x) \longrightarrow$ goes to F.

$S(x) \longrightarrow$ goes to S.

$\boxed{\text{cont}}$

0.11

→ Committor function : Hits B before A

$$f(x) = \frac{1}{(2\pi)^{d/2}\,\sigma^d} \exp\left(-\frac{\|x - x_A\|^2}{2\sigma^2}\right) \qquad \sigma = \frac{R_A}{\sqrt{d}}$$

P5.

$$d = 1 \rightarrow \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad \sigma = R_A$$

P5 // original

$$\frac{1}{2\pi\,\sigma^2} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)$$

Review.

03/15/2024.

- $A_1 \phi_{xx} + B_1 \phi_{xy} + C_1 \phi_{yy} + a\phi_x + b\phi_y + c\phi + d = 0$

$\Delta = B_1^2 - 4A_1C_1$ $\begin{cases} \Delta = 0 : \text{parabolic} \\ \Delta > 0 : \text{hyperbolic} \\ \Delta < 0 : \text{elliptic.} \end{cases}$ (※ ± 38.)

Q) what is linearity ?
$\mathcal{L}(\phi) = 0$.
Then, $\mathcal{L}(c_1\phi_1 + c_2\phi_2) = c_1\mathcal{L}(\phi_1) + c_2\mathcal{L}(\phi_2)$.

How to solve ?	Linear	Non-Linear
Characteristics (can't solve hyperbolic)	O	O
S.O.V. (can't solve non-linear)	O	X
Integral transformation	O	△ (you can integrate, may not be able to invert)
Similarity sol.	O	O

Similarity sol. — Requirement : sake of dimensions.

* Characteristics method only for "Hyperbolic" PDEs. (must when info. travels on finite domain)
e.g.) for parabolic, $dx/dt = 0$.

2 Burgers' equation
How would you parameterize
Right Now
How to parameterize.
3 Method of Characteristics
How to nondimensionalize

- 1) Characteristics.

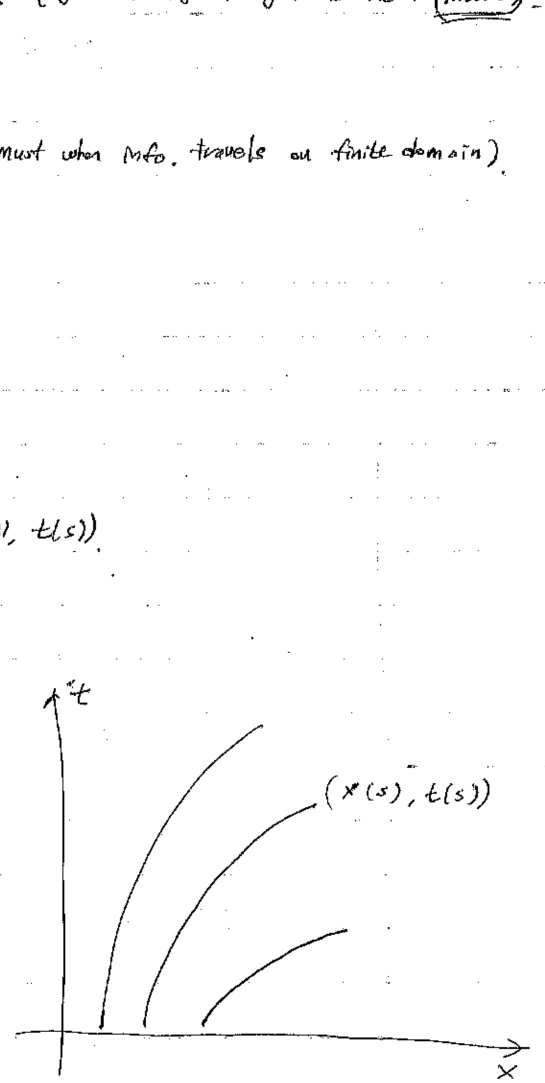
$$A\, \partial\phi/\partial t + B\, \partial\phi/\partial x + C(\phi) + D = 0.$$

$\Rightarrow \phi(x,t)$ can be parameterized as, $\longrightarrow \phi(x(s), t(s))$.

$$\Rightarrow \partial\phi/\partial s = \partial\phi/\partial x\,(\partial x/ds) + \partial\phi/\partial t\,(\partial t/\partial s)$$

$$\Rightarrow \partial\phi/\partial t + \boxed{\frac{B}{A}\frac{\partial\phi}{\partial x} + \frac{C}{A}\phi + \frac{D}{A} = 0.}$$

$$\begin{cases} \partial x/\partial s - \frac{B}{A}\,\partial t/\partial s = 0 \\ \partial\phi/\partial s = -\frac{C+D}{A}\,\partial t/\partial s. \end{cases}$$



choose $t = s \rightarrow \partial t/\partial s = 1$

$$dx/ds = \frac{B.}{A} \qquad d\phi/ds = -\frac{C+D}{A} \longrightarrow \text{Two ODEs}$$

~~Tentrad~~

Hormed E.g.) Burger's eq. $\partial u/\partial t + u\, \partial u/\partial x = 0$

$dx/dt|_{\xi} = u \qquad du/dt|_{\xi} = 0.$

depends on initial condition of this

Harmed.

- Separation of variables. — Must be Linear.

Dependent variable: u

Independent variable: x, y $\qquad u = X(x)\cdot Y(y)$.

- Similarity (scaling property)

PDE → ODE(η).

$\eta = \dfrac{x}{At^n} \qquad \tilde{T} = B t^m F(\eta)$

↓

PDE (η, t)

↓

ODE in $F(\eta)$

Use B.C to n, m

$\eta = \dfrac{x}{2\sqrt{\alpha t}}$

|||

Fourier transforms

Connected!