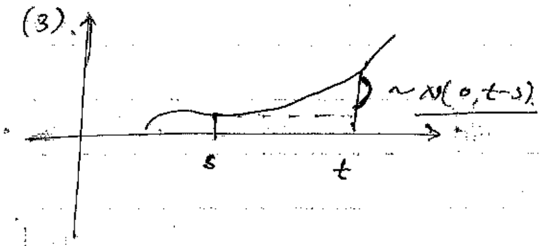
01/09/2024.

- Brownian motion.

$\{B_t,\ t \geq 0\}$: collection of random variables with

1) $B_0 = 0$
2) B_t has independent increments.
3) $B_t - B_s$ $(t > s)$ are Gaussian $\sim N(0, t-s)$

(3). 

Consequence) B.M. is continuous but not differentiable.

(2) $t=0 \quad t_1 \quad \cdots \quad t_N$

$\{B_{t_{k+1}} - B_{t_k}\}$ $k = 0, 1, \cdots, N-1$ are independent random variable!

$\Rightarrow B$ is not differentiable

Q) What determines a continuous time stochastic process X_t, $t \geq 0$ $(0 \leq t \leq T)$

F.D.D. (Finite dimension distribution)

A) It means, $t_0 = 0 \quad t_1 \quad \cdots \quad t_N = T$

Joint probability of $X_{t_0}, X_{t_1}, \ldots, X_{t_N}$ is given. $\Leftrightarrow$ $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots$

- Martingales.

(Ω, F) $\quad$ F is σ-algebra of subset of Ω

$P : (\Omega, F)$ are assignments of $A \in F \rightarrow 0 \leq P(A) \leq 1$

if $A_1, A_2, \cdots, A_N \cdots \in F$, and $A_i \cap A_j = \phi$

$$\Rightarrow \sum_{j=1}^{\infty} P(A_j) = P\left(\bigcup_{j=1}^{\infty} A_j\right)$$

if we have $X : \Omega \rightarrow \mathbb{R}$, $F_X(x) = P(\omega \in \Omega \mid X(\omega) \leq x)$ $\quad x \in \mathbb{R}$

Question : Given a consistent set of probabilistic distribution of process X_t, $0 \leq t \leq T$. Does there exist Ω and F and function $X_t(\omega) : \Omega \rightarrow \mathbb{R}^d$ $0 \leq t \leq T$ such that F.D.D. of X_t (by Kolmogorov continuity ~)

$\rightarrow$ We need that the F.D.D. has stochastic continuity.

$$\Rightarrow \lim_{h \to 0} \sup_{0 \leq t \leq T} P(|X_{t+h} - X_t| > \delta) = 0 \quad \text{for all } \delta > 0 \Leftrightarrow \text{Continuity "in probability"}$$

- Discrete time Martingale.

$\{X_n(\omega)\}$ $n=0,1,\cdots,$ $\omega\in\Omega$

$E[X_n] = \int_\Omega X_n(\omega)\,dP(\omega)$ / $F_n \subset F$, $F_n\subset F_{n+1}$, $F_n = \sigma\{X_0, X_1, \cdots, X_n\}$

$E[X_n \mid \underbrace{F_{n-1}}_{\text{information up to } n-1 \text{ (given)}}] = X_{n-1}$ $n=1,2,\cdots$

E.g.) suppose $X_1, X_2, \cdots, X_n, \cdots$ ind. dist. and $E[X_j]=0$.
assume $S_n = X_1 + \cdots + X_n$,

$$\Rightarrow E[S_n \mid F_{n-1}] = E[X_1+\cdots+X_n \mid X_0, X_1, \cdots, X_{n-1})$$

$$= X_1 + \cdots + X_{n-1} + E[X_n \mid X_0, \cdots, X_{n-1}]$$
$$= X_1 + \cdots + X_{n-1} + \cancel{E[X_n]}^{0}$$
$$= X_1 + \cdots + X_{n-1} = S_{n-1} \Rightarrow E[S_n \mid F_{n-1}] = S_{n-1}$$

Ex ① Note that, $B_{t_N} = (B_{t_N} - B_{t_{N-1}}) + (B_{t_{N-1}} - B_{t_{N-2}}) + \cdots + (B_{t_1} - B_{t_0})$

Clearly, B_{t_N} is ① sum of independent ② expectations of zero.

$\Rightarrow$ B_{t_N} is Martingale

Ex ② What about $B_t^2 - t$ $\rightarrow$ show that $E[(B_t^2 - t \mid F_s] = B_s^2 - s$

$\Rightarrow$ show $E[B_t^2 - B_s^2 \mid F_s] = t-s$

$$B_t^2 - B_s^2 = (B_t - B_s)^2 + 2B_t\cdot B_s - B_s^2 - B_s^2 = (B_t - B_s)^2 + 2(B_t - B_s)\cdot B_s$$

$$\Rightarrow E[B_t^2 - B_s^2 \mid F_s] = \underbrace{E[(B_t-B_s)^2 \mid F_s]}_{=\text{Variance} = t-s} + \underbrace{2E[(B_t-B_s)B_s \mid F_s]}_{=2\cdot \cancel{E[(B_t-B_s)\mid F_s]}^{0}\cdot B_s}$$

$$= t-s$$

Ex ③ $e^{\alpha B_t - \alpha^2 t/2} = M_t^{(\alpha)} = $ Martingale.

- Brownian motion - quadratic 01/11/2024

$\{B_t, t \geq 0\} \rightarrow \begin{cases} 1 & B_0 = 0 \\ 2 & B_t \text{ has ind. increments} \\ 3 & B_t - B_s \sim N(0, t-s) \end{cases}$ Q) Can we define on continuous functions?

$\Rightarrow$ "Continuity" of B_t is implied in 1, 2, 3.

- B_t is a Martingale.

Suppose (Ω, F) is prob. space, $B_t \sim B_t(\omega)$, $\omega \in \Omega$ is defined,

$B_t(\omega) : \Omega \to \mathbb{R}$, $t \geq 0$.

$\{\omega \mid B_t(\omega) \leq x\} \in F_t \subset F$

↳ σ-algebra (information) involving path up to t

Note : F_t is generated by events $\{\omega \in \Omega \mid B_{t_1}(\omega) \leq x_1, \cdots, B_{t_N}(\omega) \leq x_n\}$
$\begin{pmatrix} \text{with } 0 \leq t_1 < t_2 \cdots < t_N \leq t \\ \text{and } x_1, x_2, \cdots, x_N \in \mathbb{R} \end{pmatrix}$

Here, Martingale property of B_t, $t \geq 0$ $\Rightarrow$ $\underline{E[B_t \mid F_s] = B_s}$ — ①

why? ① $\equiv E[B_t - B_s \mid F_s] = 0$ (True, since $B_t - B_s \sim N(0, t-s)$ Gaussian).

E.g.) 1) $E[|B_t|] = \int_{-\infty}^{\infty} |x| \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} dx < \infty$ (finite)

2) $E[B_t^p] = \int_{-\infty}^{\infty} x^p \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} dx < \infty$ (for all p even) (finite)
$= 0$ (p is odd)

3) $E[e^{\alpha B_t}] = \int_{-\infty}^{\infty} e^{\alpha x} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} dx = e^{\alpha^2 t/2}$

$\Rightarrow$ we conclude $\underset{\underset{M_t^\alpha}{\parallel}}{e^{\alpha B_t - \alpha^2 t/2}}$ is a Martingale.

show $\Rightarrow$ $E[M_t^\alpha \mid F_s] = M_s^\alpha \Rightarrow \underline{E[e^{\alpha B_t - \alpha^2 t/2} \mid F_s] = e^{\alpha B_s - \alpha^2 s/2}}$ Random

$\Leftrightarrow E[e^{\alpha B_t - \alpha B_s} \mid F_s] = E[e^{\alpha(B_t - B_s)} \mid F_s] = e^{\alpha^2(t-s)/2}$ $(B_t - B_s)$ is Gaussian Variable

$\Leftrightarrow E[\, e^{\alpha(B_t - B_s)} \mid F_s] = e^{\alpha^2 (t-s)/2}$

$\Rightarrow$ We can get rid of F_s ($\because$ independent increments.)

- Total variation / Quadratic variation.

$f(t)$, $0 \le t \le T$ is a given function. $t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad t_N = T$

Partitions ($t_0 \sim t_N$) is called Π_N

$\Rightarrow$ Π_N : $\max_j |t_{j+1} - t_j| \to 0$ as $N \to \infty$

Define $TV_T(f) = \lim_{N\to\infty} \sup_{\Pi_N} \sum_{j=0}^{N-1} |f(t_{j+1}) - f(t_j)|$.

$\to$ Any f such that $\max_t |f'(t)| < \infty \Leftrightarrow TV_T(f) \le \lim_{N\to\infty} \sup_{\Pi_N} \sum_{j=0}^{N-1} f'(t_j) \, |t_{j+1} - t_j|$

$\Rightarrow TV_T(f) \le \max |f'(t_j)| \lim_{N\to\infty} \sup_{\Pi_N} T = \max_t |f'(t)| \, T$.

$\Rightarrow$ If it's differentiable, $\to TV_T(f) < \infty$ (finite).

However, if f is ~~incr~~ monotone (increasing),

$$TV_T(f) = \lim_{N\to\infty} \sup_{\Pi_N} \sum_{j=0}^{N-1} \left(f(t_{j+1}) - f(t_j)\right) = f(T) - f(0) < \infty$$

• Riemann integrale.

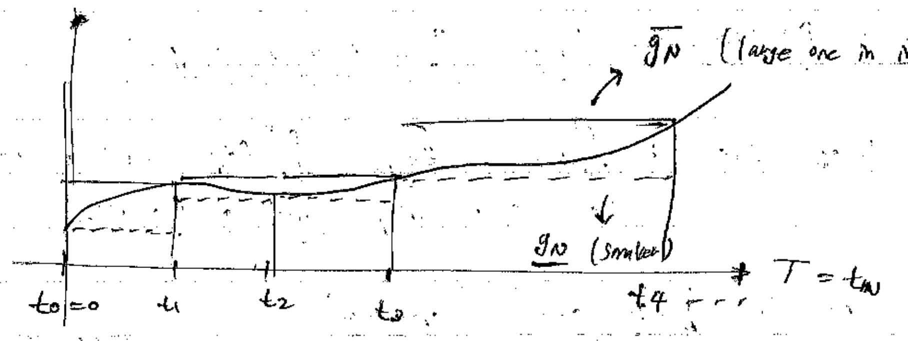
$$\int_0^T g(s)\, df(s) \doteq \lim_{N\to\infty} \sup_{\Pi_N} \sum_{j=0}^{N-1} g(t_j^*) \{ F(t_{j+1}) - F(t_j) \}$$

$\underset{\text{Bounded Continuous (BC)}}{g(s)}$ ↓, $\underset{\text{Bounded total variation (BV)}}{df(s)}$ ↓

$t_j^* \in (t_j ; t_{j+1})$

$$\left| \int_0^T g(s)\, df(s) \right| \le \max_{0\le s\le T} |g(s)|\ TV_T(f)$$

Hint: if continuous → expressible in <u>step function</u>.



$$\Rightarrow \underline{g_N}(t) \le g(t) \le \overline{g_N}(t) \Rightarrow \limsup_{N\to\infty}\ \sup_{0\le t\le T} |\overline{g_N}(t) - \underline{g_N}(t)| = 0$$

Difference goes to zero.

for g step function w.r.t. Π_N

$$\Rightarrow \int_0^T g\, df = \sum_{j=0}^{N-1} g(t_j^*) (f(t_{j+1}) - f(t_j)) \ ..$$

$$\Rightarrow \sum_{j=0}^{N-1} \underline{g}(t_j^*) | f(t_{j+1}) - f(t_j) | \le \int_0^T g\, df \le \sum_{j=0}^{N-1} \overline{g}(t_j^*) | f(t_{j+1}) - f(t_j) |$$

→ <u>Sandwiched!</u>

BUT, $TV_T(B_t) = +\infty \quad (w.p.\ 1)$ Brownian motion...

<u>Note</u>

$$E\left[\sum_{j=0}^{N-1} |B_{t_{j+1}} - B_{t_j}| \right] = \sum_{j=0}^{N-1} E\{ |B_{t_{j+1}} - B_{t_j}| \}$$

$$\hookrightarrow \int_{-\infty}^{\infty} |x| \frac{e^{-x^2/(t_{j+1}-t_j)}}{\sqrt{2\pi(t_{j+1}-t_j)}}\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} |y| \frac{e^{-y^2}}{\sqrt{2\pi}}\, dy}_{=c} \cdot \sqrt{t_{j+1}-t_j}$$

$$= c \cdot \sum_{j=0}^{N-1} \sqrt{t_{j+1}-t_j}$$

Suppose Π_N is even partition $\to$ $t_{j+1} - t_j = \Delta t$; $N\Delta t = T$

(just show divergence for particular case)

$$\Rightarrow \; c \sum_{j=0}^{N-1} \sqrt{t_{j+1}-t_j} = cN\sqrt{\Delta t} = CT \cdot \frac{1}{\Delta t} \to \infty$$

{ Using Borel – Cantelli (B.C.) lemma, we show that
$TV_T(B_t) = +\infty$ w.p.1. }

$\Rightarrow$ Conclusion : $\int_0^T f(B_s)\, dB_s$ "cannot be" a Riemann integral.

$\to$ New theory (Itô) : $\int_0^T g\, df = gf\Big|_0^T - \int_0^T f dg$, will this work?

$\to$ No! since B is not differentiable.

But, B_t has $\boxed{\text{finite quadratic variation}}$

$$\limsup_{\substack{N\to\infty \\ \Pi_N}} \sum_{j=0}^{N-1} |B_{t_{j+1}} - B_{t_j}|^2 = T \quad \text{in mean square.}$$

$$E\left\{ \sum_{j=0}^{N-1} (B_{t_{j+1}} - B_{t_j})^2 - T \right\}^2 \xrightarrow[\text{for any } \Pi_N]{N\to\infty} 0 \quad \text{(its what mean square means)}$$

$$E\left\{ \sum_{j=0}^{N-1} \underbrace{\left[|B_{t_{j+1}} - B_{t_j}|^2 - (t_{j+1} - t_j) \right]}_{\text{Random Variables and mean} = 0} \right\}^2 = E\left(\sum_{j=0}^{N-1} [\quad]^2 \right)$$

Note: no cross terms survive ($\because$ independent)

$$= \sum_{j=0}^{N-1} E\left\{ (B_{t_{j+1}} - B_{t_j})^4 - 2(B_{t_{j+1}} - B_{t_j})^2 (t_{j+1} - t_j) + (t_{j+1} - t_j)^2 \right\}$$

(Fact : if X is Gaussian $\to$ mean $= 0$, $E[X^4] = 3 (E(X^2))^2$)

$$= \sum_{j=0}^{N-1} \left[3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 \right]$$

$$= 2 \sum_{j=0}^{N-1} (t_{j+1} - t_j)^2 \le 2 \cdot \max_j (t_{j+1} - t_j) \cdot T \xrightarrow[N\to\infty]{} 0$$

$$\therefore \; QV_T(B) = T$$

$$\sum_{j=0}^{N-1} \left(B_{t_{j+1}} - B_{t_j}\right)^2 - T$$

for simple $N \Delta t = T$, $\frac{1}{\sqrt{\Delta t}} \cdot \left\{ QV_T^N(B) - T \right\} \longrightarrow N(0, 2T)$ #

01/16/2024

- Kolmogorov continuity. (B_t is continuous w.p. 1)

$\Omega = C([0,T];\mathbb{R})$ $\quad \mathcal{F}_T = \sigma$-algebra generated by cylinder set.

$\rightarrow 0 \le t \le T \quad \mathcal{F}_t = \sigma\{B_s, s \le t\}$

cylinder set : $\{\omega \in \Omega \mid B_{t_1}(\omega) \le x_1, \cdots, B_{t_N}(\omega) \le x_n\}$

↳ depends on $\pi_N, x_1, x_2, \cdots, x_n$

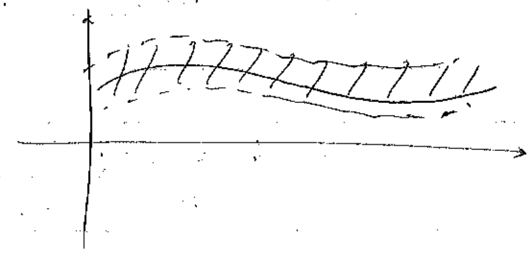
$\langle$ Note: $\omega \in \Omega$, $\omega(t) = B_t(\omega) \rangle$

↳ Path of Brownian motion at time t.

$(\Omega, \mathcal{F}_{0\le t\le T}, P) \rightarrow P$ makes paths to follow rules of Brownian motion.

Ω is also a metric space. : $dist(\omega_1, \omega_2) = \max_{0\le t\le T} \| \omega(t_1) - \omega(t_2) \|$

Define open set : $\mathcal{O}_0 = \{\omega_1 \mid d(\omega_0, \omega_1) < \delta\} \Rightarrow$



- Elementary def. of B.M. provides a prob. law on any cylinder sets of Ω.

$\Rightarrow$ if $A \in \mathcal{F}_T$ and is a cylinder set, then $P(A)$ is defined.

since joint law of $B_{t_0}(\omega), B_{t_1}(\omega), \cdots, B_{t_N}(\omega)$ is Gaussian

$B_{t_0} = 0, B_{t_1} - B_{t_0}, \cdots, B_{t_N} - B_{t_{N-1}}$ are independent Gaussian

E.g.) $B_{t_2} = \underbrace{(B_{t_2} - B_{t_1}) + (B_{t_1} - B_{t_0})}_{\text{sum of Gaussians}} = B_{t_2}$

Ex)

- Kolmogorov continuity. (K.C.)

$\Rightarrow$ Suppose P (any P on $(\Omega, \mathcal{F}_t\ _{0\le t\le T})$) has property that there exist α, β, C positive.

$$E^P\{|X_t(\omega) - X_s(\omega)|^\alpha\} \le C|t-s|^{1+\beta} \quad \underline{0 \le s < t \le T}$$

Then, P extends to prob. law on all of $(\Omega, \mathcal{F}_T)$

Logic : Cylinder set (satisfied) $\xrightarrow{K.C.}$ all set (satisfied).

For B.M.,

$$E\{|B_t - B_s|^\alpha\} = \int_{-\infty}^{\infty} |x|^\alpha \frac{e^{-\frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\,dx \qquad \left(x/\sqrt{t-s} = y \right)$$

$$\Rightarrow E\{|B_t - B_s|^\alpha\} = C\alpha\,(t-s)^{\alpha/2} \quad \rightarrow \alpha/2 > 1 \quad \text{(Kolmogorov criteria)}.$$

$$\Rightarrow \alpha > 2$$

$\alpha = 4,\ \beta = 1 \rightarrow$ Kolmogorov criterion works.

< Inequality > (not continuous in general)

Suppose $f(t)$ $0 \le t \le 1$, s.t. $\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{(x-y)^{\alpha p+1}}\,dx\,dy < \infty \quad (\alpha p > 1)$

Then, GRR (1972) $\Rightarrow |f(t) - f(s)| \le C_{\alpha,p} |t-s|^{\alpha - \frac{1}{p}} \left(\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p+1}}\,dx\,dy \right)^{1/p}$ — ①

$$\left(\frac{\alpha p - 1}{p} > 0 \quad \text{(Hölder exponents)} \right)$$

Intuitively as $x \approx y$ $f(x) \approx f(y)$ to make ① integrable.

↳ so it kind of implies the continuity.

⇒ Apply this inequality (①) with Kolmogorov criterion,

→ let $u(h,\omega) = \max_{|t-s| \le h} |X_t(\omega) - X_s(\omega)|$ a decreasing function of h.

GRR ⇒ $u(h,\omega) \le C_{\alpha,p}\, h^{\alpha - 1/p}\, \tilde{B}(\omega)^{1/p}$ where $\tilde{B}(\omega) = \int_0^1\int_0^1 \frac{|X_t(\omega) - X_s(\omega)|^p}{|t-s|^{\alpha p+1}}\,dt\,ds$

By K.C., $E(\tilde{B}(\omega)) < \infty$ (finite)

(Recall K.C.

$E(|X_t - X_s|^p) \le |t-s|^{1+\beta} \Rightarrow \underbrace{1 + \beta - (\alpha p + 1) < 1}_{\text{suitable parameters}} \quad \left(\because \int \frac{1}{x^p} \text{ converges } p > 1 \right)$)

$P(\omega \in \Omega \mid \tilde{B}(\omega) < \infty) = 1$

→ P extends to whole. (K.C.)

$(\Omega, \mathcal{F}_{0\le t\le T}, P)$, $\Omega = C([0,T];\mathbb{R})$ $\begin{pmatrix} P - \text{B.M. law} \\ B_t(\omega) \sim \text{B.M.} \end{pmatrix}$

and $\forall \omega \in \Omega$ $f(t,\omega)$ $[0,T] \times \Omega \to \mathbb{R}$

$\Rightarrow$ define $\int_0^T f(s,\omega)\, dB_s(\omega) \sim$?

- f is non-anticipating : $\{\omega \in \Omega \mid f(t,\omega) \in A\} \in \mathcal{F}_t$ $\begin{pmatrix} 0\le t\le T \\ A \subset \mathbb{R} \end{pmatrix}$.

- $E^P \int_0^T f^2(s,\omega)\, ds < \infty$: Ito integral. $\int_0^T f(s,\omega)\, dB_s(\omega)$

$$= \lim_{N\uparrow\infty} \sum_{k=0}^{N-1} f(t_k,\omega) \underline{\left(B_{t_{k+1}}(\omega) - B_{t_k}(\omega)\right)}$$

(There is a direction of time) $\leftarrow$ (increasing increments).

Any situation = Forward int. + Backward int.

E.g.) $X_t(\omega) \to$ Martingale (continuous function of ω)

$X_t^2(\omega) - \int_0^t g_s^2(\omega)\, ds$ is Martingale, then, $X_t(\omega) = \int_0^t g_s(\omega)\, dB_s(\omega)$

Ito Stochastic integral.

$\Rightarrow$ Holds for arbitrary functions!

$\rightarrow$ Preview Chapter 3.

$B_t(\omega)$ = mart.

$B_t^2(\omega) - t$ = mart.

Stochastic integrals. 01/18/2024.

Brief review.

$$\begin{cases} \Omega = C([0,T];\mathbb{R}) & \text{: continuous.} \\ w_1, w_2 \in \Omega & \text{: paths.} \\ d(w_1, w_2) = \max_{0\le t\le T} |w_1(t) - w_2(t)| \end{cases}$$

for $0 \le t \le T$ $\mathcal{F}_t$ = σ-algebra generated by $\beta_s(\omega)$ $s \le t$

Notation : $\beta_t(\omega) = \omega(t)$, in general, $f : [0,T] \times \Omega \to \mathbb{R} \Rightarrow f(t,\omega)$

with this notation, $\beta(t,\omega) = \beta_t(\omega) = \omega(t)$ ——— $(*)$.

→ In general, Ω can be considered as subspace of set of all functions on $[0,T]$

E.g.) $\tilde{\Omega}$ = set of real value functions on $[0,T]$ → $\Omega \subset \tilde{\Omega}$

$\mathcal{F}_t$ can be defined on $\tilde{\Omega}$ as well.

Given a set of finite dimension distribution, $F_{t_1,\dots,t_n}(x_1,\dots,x_n)$ $0 \le t_1 \cdots \le t_n \le T$.

⇒ $P(\beta_{t_1}(\omega) \le x_1, \dots, \beta_{t_n}(\omega) \le x_n)$ can be associated uniquely with a probability law on $\tilde{\Omega}$ assuming only that they are consistent

Q) How to assign prob. on $\sup_{0\le s\le t} \beta_s(\omega) = M_t(\omega)$ → we don't know

If however, P is stochastic continuous. $\sup_{0\le s\le t\le T} P(|\beta_t(\omega) - \beta_s(\omega)| > \delta) \longrightarrow 0$ as $t \to s$, (for any $\delta > 0$)

Then, any countable dense set in $[0,T]$ determines probabilities on $[0,T]$

We can also define what we mean $f(t,\omega)$ $\omega \in \Omega$ $t \in [0,T]$ being integrable.

Suppose Kolmogorov continuity holds $(K.C.)$: $E[|\beta_t(\cdot) - \beta_s(\cdot)|^{\alpha}] \le C_T |t-s|^{1+\beta}$ $\alpha, \beta, C_T > 0$

Then, with probability 1, the continuous function in $\tilde{\Omega}$, that is, the set $\Omega \subset \tilde{\Omega}$ have probability 1, $P(\Omega) = 1$.

(Drop $\tilde{\Omega}$ completely!)

• Stochastic integrals. $\{\Omega = C([0,T];\mathbb{R})\ \ \mathcal{F}_t,\ 0\le t\le T\ ,\ P = \text{Brownian motion law}.\}$

→ If $f(t,\omega) : [0,T]\times\Omega \to \mathbb{R}$, we assume that,

① $E\int_0^T f^2(t,\omega)\,dt < \infty$

② non-anticipating → $\{\omega\in\Omega \mid f(t,\omega)\in A\} \in \mathcal{F}_t$ $(A\subset\mathbb{R})$ for all $0\le t\le T$

→ Partition :

$t_0=0\ \ t_1\ \ \ldots\ \ \ldots\ \ t_N = T$

Define : $\int_0^T f(t,\omega)\,dB_t(\omega) = \underset{\text{Mean square}}{\lim} \sum_{j=0}^{N-1} f(t_j,\omega)\left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right)$

Note : Increment is always forward.

⇒ Use approx. of $f(t,\omega)$ called simple function.

$$f(t,\omega) = \sum_{k=0}^{N-1} e_k(\omega)\,\chi_{[t_k,t_{k+1}]}(t) \qquad e_k(\omega)\in\mathcal{F}_{t_k}$$

$$I_T(f) = \int_0^T f(s,\omega)\,dB_s(\omega) = \sum_{k=0}^{N-1} e_k(\omega)\{B_{t_{k+1}}(\omega) - B_{t_k}(\omega)\}$$

For simple function following prop hold ⇒ ① Additivity : $\int_0^T f\,dB = \int_0^u f\,dB + \int_u^T f\,dB\ (0\le u\le T)$

② Linearity : $\int_0^T (f+g)\,dB = \int_0^T f\,dB + \int_0^T g\,dB$.

③ $E\left[\int_0^T f\,dB\right] = 0$,

(*) : Iterative conditional expectation.

$$= E\left[\sum_{k=0}^{N-1} e_k(\omega)\left(B_{t_{k+1}}(\omega) - B_{t_k}(\omega)\right)\right]$$

$$= \sum_{k=0}^{N-1} E\{e_k(\omega)\left(B_{t_{k+1}} - B_{t_k}\right)(\omega)\} \quad (*)$$

$$= \sum_{k=0}^{N-1} E\{E[e_k(B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_{t_k}]\}$$

$$= \cancel{\sum_{k=0}^{N-1} (\,)\{E\{e_k \mid \mathcal{F}_{t_k}\}\ (B_{t_{k+1}} - B_{t_k})\}}$$

$$= \sum_{k=0}^{N-1} E[e_k\ \underbrace{E\{B_{t_{k+1}} - B_{t_k} \mid \mathcal{F}_{t_k}\}}_{=0\ (\text{independent})}] = 0.$$

Property ④ : Ito isometry : $E[I_T^2(f)] = \int_0^T E\{f^2(t,\cdot)\}\,dt < \infty$

$$E\{(I_T(\omega))^2\} = E\left\{\left(\sum_{k=0}^{N-1} e_k (B_{t_{k+1}} - B_{t_k})\right)^2\right\}$$

$$= E\left[\sum_{k=0}^{N-1}\sum_{k'=0}^{N-1} e_k e_{k'} (B_{t_{k+1}} - B_{t_k})(B_{t_{k'+1}} - B_{t_{k'}})\right]$$

$$= \sum_k \sum_{k'} E\{\underbrace{e_k e_{k'} (B_{t_{k+1}} - B_{t_k})(B_{t_{k'+1}} - B_{t_{k'}})}\}$$

$$= \sum_k \sum_{k'} E\{E\{\text{———} \mid F_{t_{k'}}\}\}$$

(assume: $k \le k'$) ↳ highest one. ($\because k \le k'$)

$$= \sum_k \sum_{k'} E[e_k e_{k'} E[\text{———} \mid F_{t_{k'}}]]$$

if $k \neq k'$ → this term is zero → diagonal terms survive.

$$\Rightarrow \sum_k E[e_k^2 (B_{t_{k+1}} - B_{t_k})^2] = \sum_k E[e_k^2](t_{k+1} - t_k) \quad (\because \text{independent})$$

$$= \int_0^T E(f^2(t,\omega))\,dt$$

$$f(t,\omega) = \begin{cases} e_k(\omega) & (t_k \le t \le t_{k+1}) \\ 0 & (\text{otherwise}) \end{cases}$$

Property ⑤ For f simple $\int_0^T f\,dB$ is an F_t martingale in $0 \le t \le T$ that is continuous in t.

($E[I_t \mid F_0] = I_0 = 0$ ~ similar to P3)

If $I_t(\omega) = \int_0^t f(s,\omega)\,dB_s(\omega) \Rightarrow E[I_t \mid F_s] = I_s$

Summary : $I_t(\omega)$ is continuous and square integrable martingale.

since, $E[I_t^2] = \int_0^t E[f^2(s,\cdot)]\,ds$.

Q) Given property 1~5 → How to complete the theory?

• Lemma (Oksendal).

Suppose $f : [0,T] \times \Omega \to \mathbb{R}$, non-anticipating and square integrable.

Without non-anticipating property, we know that there always exists a sequence of simple $f_n(t,w)$ s.t. $E\int_0^T \left(f(t,w) - f_n(t,w)\right)^2 dt \longrightarrow 0$ $(n \uparrow \infty)$

→ If also f is non-anticipating then $f_n(t,w)$ can also be chosen to be non-anticipating. (intuitive).

↓

Suppose this is known → How to complete?

→ How do we extend property ⑤ to any f that is non-anticipating and square integrable.

$\int_0^t f_n(s,w)\,ds = I_n^t(w)$ is a continuous square integrable F_t

Review convergence

(1) X_n R.V. $n = 1, 2, \cdots$, $X_n \to X$ in Mean square. if $E(X_n - X)^2 \to 0$

(2) $X_n \to X$ in probability if $P(|X_n - X| > \delta) \to 0$ as $n \uparrow \infty$ for $\delta > 0$

(3) $X_n \to X$ a.s if $X_n(w) \to X(w)$ for a set of $w \in \Omega$ that has prob 1.

→ Suppose we show that $P\left\{\max_{0 \le t \le T}\left(\left|I_n(t,w) - I_m(t,w)\right| > \delta\right\}\right. \longrightarrow 0$

as $n, m \to \infty$ for any $\delta > 0$.

(In words, $I_n(t,w)$ is a Cauchy-sequence uniformly in t, $0 \le t \le T$ in probability with respect to (w, n, t) (w).)

Suppose X_n is Cauchy in M.S (a.s) → X $\underline{\text{conv. prob}}$ exists limit

" X_n " " in prob

metrics

conv. in prob. $\equiv E\left(\dfrac{|X_n - X|}{|X_n - X| + 1}\right) \to 0$ as $n \uparrow \infty$

⇒ $I_n(t,w) \xrightarrow[\text{is Cauchy}]{} $ uniformly continuous

$P\left\{\sup_{0 \le t \le T} |I_n(t,w) - I_m(t,w)| > \delta\right\} \underset{n,m\to\infty}{\longrightarrow} 0$ → Continuous!

- The stochastic integral 01/23/2024

→ Main fact : $\begin{cases} \text{Integrand } f(t,\omega) : [0,T] \times \Omega \to \mathbb{R}, \quad \Omega = C([0,T];\mathbb{R}) \\ \mathcal{F}_t,\ 0 \le t \le T, \quad P : \text{makes path B.M.} \end{cases}$

→ Notation : $\omega(t)$ $0 \le t \le T$ is decided by $B_t(\omega) = \omega(t)$.

→ Properties :
- $E\int_0^T f(s,\cdot)\,ds < \infty$
- Non-anticipating $\{\omega \in \Omega \mid f(t,\omega) \in A\} \in \mathcal{F}_t$, $0 \le t \le T$. $A \subset \mathbb{R}$
- simple functions

$$f(t,\omega) = \sum_{k=0}^{N-1} e_k(\omega)\,\chi_{[t_k, t_{k+1})} \qquad e_k(\cdot) \in \mathcal{F}_{t_k}$$

$$\Updownarrow$$

$$\{\omega \in \Omega \mid e_k(\omega) \le x\} \in \mathcal{F}_{t_k} \quad \forall x \in \mathbb{R}$$

⇒ For simple f, $\displaystyle\int_0^T f(s,\omega)\,dB_s(\omega) = \sum_{k=0}^{N-1} f_k(\omega)\{\underbrace{B_{t_{k+1}}(\omega) - B_{t_k}(\omega)}_{\text{Adding ahead (forward)}}\}$

① Additivity.

② Linearity.

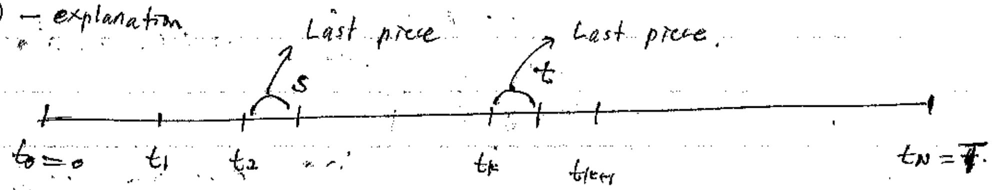
③ $E\left(\int_0^T f\,dB\right) = 0$

④ $E\left[\left(\int_0^T f\,dB\right)^2\right] = E\int_0^T f^2\,ds$

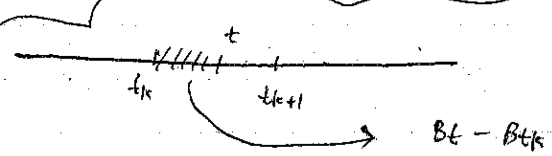
⑤ $I_t(\omega) = I(t,\omega) = \int_0^t f(s,\omega)\,dB_s(\omega) \quad \left(= \int_0^T \chi_{[0,t)}\, f(s,\omega)\,dB_s(\omega)\right)$.

↳ $I_t(\omega)$ is continuous in t, square integrable, $\mathcal{F}_t$ martingale. $(E[I_t \mid \mathcal{F}_s] = I_s)$ $0 \le s \le t \le T$.

⑤ – explanation



$$\left.\begin{aligned} I_s &= \sum_{k'=0}^{n_s - 1} e_{k'}(B_{t_{k'+1}} - B_{t_{k'}}) + e_{n_s - 1}(B_s - B_{t_{n_s - 1}}) \\ I_t &= \quad\quad \text{''} \quad\quad + e_{n_t - 1}(B_t - B_{t_{n_t - 1}}), \end{aligned}\right\}$$



From inherited continuity of B.M. → I_t is continuous

Pass the limit from simple f to general f.

Oksendal lemma

General approximation theorem.

$f(t,\omega)$ sq. int. $E\int_0^T f^2(s,\cdot)\,ds < \infty \Rightarrow \exists\, f_n(t,\omega)$ simple s.t. $E\int_0^T (f-f_n)^2 \longrightarrow 0$ as $n \to \infty$

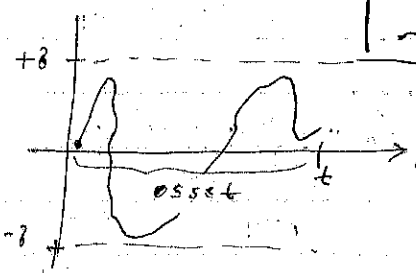
< Additionally, f_n is non-anticipating >

$\boxed{\text{Kolmogorov inequality}}$ → Let M_t be integrable martingale $\Rightarrow E\{|M_t|^p\} < \infty$, $1 \le p < \infty$

$E[M_t | F_s] = M_s$

assume M_t continuous, then $\boxed{P\{\max_{0\le s\le t} |M_s| > \delta\} \le \frac{1}{\delta^p} E\{|M_t|^p\}}$

↓



Probability of exiting the bound

→ Apply!

Let $I_n(t,\omega) = \int_0^t f(t,\omega)\,dB_s(\omega)$ $(T,\cdot)$

K.I.

$$\Rightarrow P\left\{\max_{0\le t\le T} |I_n(t,\omega) - I_m(t,\omega)| > \delta\right\} \le 1/\delta^2 \cdot E\left\{\left(I_n(T,\cdot) - I_m(T,\cdot)\right)^2\right\}$$

$$= 1/\delta^2\, E\left\{\left(\int_0^T (f_n(s,\cdot) - f_m(s,\cdot))\,dB_s\right)^2\right\}$$

$$= 1/\delta^2\, E\left[\int_0^T (f_n(s,\cdot) - f_m(s,\cdot))^2\,ds\right]$$

Itô's Isometry

$\to 0$ $n,m\to\infty$ $\because f_n \to f$ in M.S.Q.

Conclusion: $I_n(t,\omega)$ is a continuous [illegible].

↳ It has unique limit $I(t,\omega)$ s.t. $P\left\{\max_{0\le t\le T} |I_n(t,\cdot) - I(t,\cdot)| > \delta\right\} \longrightarrow 0$ as $n\to\infty$

→ we can prove using Borel-Cantelli lemma → find subsequence $I_{n_k}(t,\omega)$ s.t.

$I_{n_k}(t,\omega) \to I(t,\omega)$ as $n_k \to \infty$ unique in t w.p. 1

We now have stochastic integral.

$$I(t,\omega) = \int_0^t f(s,\omega)\, dB_s(\omega)$$

with all 5 properties of the S.I. for simple f.

① Additivity
② Linearity
③ Mean zero.
④ Ito isometry
⑤ $I(t,\omega)$ is a cont. sq. int. Mart.

(0) Note: $I(t,\omega) = \int_0^t f(s,w)\, dB_s(\omega) \quad 0 \le t \le T$ on $(\Omega, \mathcal{F}_t\ 0 \le t \le T, P)$

is cont. sq. integrable Martingale.

(1) Converse : Every cont. sq. int. Martingale is a Brownian stochastic integral.

(2) Small converse : If M_t is cont. sq. int. Mart. s.t. its quadratic variation is t

then M_t is a Brownian motion. → Doesn't have to be original!

why? → show by converse

$M_t(\omega) = \int_0^t f(s,\omega)\, dB_s(\omega)$ But $E(M_t^2) = E\int_0^t f^2 ds = t$

In addition, ∵ For any S. integral $I(t,\omega) = \int_0^t f(s,\omega)\, dB_s(\omega)$

$\Rightarrow QV(I) = \int_0^T f^2(s,\omega)\, ds$ (∵ Ito isometry)

so $QV_t(M) = t \quad (0 \le t \le T) \Rightarrow \int_0^t f^2(s,\omega)\, ds = t \Rightarrow f^2 = 1$

comment) let $X(t,\omega) = \int_0^t f(s,\omega)\, dB_s(\omega)$ and suppose $|f^2(s,\omega)| = 1$ $(0 \le t \le T)$

$\Rightarrow X(t,\omega)$ is a Brownian motion that's not $B(t,\omega)$

Small converse is proved!

QV. proofs.

01/23/2024.

Ito's formula. (p24 ~ p.25 in lecture notes.).

$g(x) : \mathbb{R} \to \mathbb{R}$. $g(x), g'(x), g''(x)$ are bounded.

Then, $g(B_t) - g(B_s) = \int_s^t g'(B_u)\,dB_u + \underbrace{\frac{1}{2}\int_s^t g''(B_u)\,du}_{\text{Ito term.}}$.

$t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad T = t_N$ (assume $\max_k |t_{k+1} - t_k| \to 0$ as $N \to \infty$)

$$g(B_T) - g(B_0) = \sum_{k=0}^{N-1} \{ g(B_{t_{k+1}}) - g(B_{t_k}) \}$$

$$= \underbrace{\sum_{k=0}^{N-1} g'(B_{t_k})(B_{t_{k+1}} - B_{t_k})}_{①}$$ (Taylor's expansion)

$$+ \underbrace{\sum_{k=0}^{N-1} \frac{1}{2} g''(B_{t_k})(B_{t_{k+1}} - B_{t_k})^2}_{②}$$ (assume g''' exists).

$$+ \underbrace{\sum_{k=0}^{N-1} \frac{1}{6} g'''(B_{t_k^*})(B_{t_{k+1}} - B_{t_k})^3}_{③}$$ $t_k^* \in [t_k, t_{k+1}]$ (mean value thm).

① $\to \int_0^T g'(B_s)\,dB_s$ ② $\to \frac{1}{2}\int_0^T g''(B_s)\,ds$ ③ $\to 0$ why?

$$\left| \sum_{k=0}^{N-1} g'''(t_k^*)(B_{t_{k+1}} - B_{t_k})^3 \right| \le \max_x |g'''(x)| \underbrace{\max_k |B_{t_{k+1}} - B_{t_k}|}_{= 0 \text{ w.p. } 1} \underbrace{\sum_{k=0}^{N-1} (B_{t_{k+1}} - B_{t_k})^2}_{\text{bounded w.p. } 1}$$

(Brownian continuity).
→ Uniform continuity.

$\therefore$ ③ $\to 0$

$$\therefore\ g(B_t) - g(B_s) = \int_s^t g'(B_u)\,dB_u + \frac{1}{2}\int_s^t g''(B_u)\,du.$$

Q) How to generalize? → continued...

$$\because\ g(B_T) - g(B_0) = \int_0^T g'(B_s)\,dB_s + \frac{1}{2}\int_0^T g''(B_s)\,ds$$

01/25/2024.

- K.I $\rightsquigarrow$ Stopping times.

- K.I. in discrete time.

$$\begin{cases} M_n, \ n \geq 0 \quad \text{Martingale at } (\Omega, F_n, 0 \leq n \leq N, P) \\ E[|M_n|^p] < \infty \quad 1 \leq p < \infty \\ E[M_n | F_k] = M_k \quad 0 \leq k \leq n < \infty \end{cases}$$

$$\Rightarrow P\{\max_{0 \leq k \leq n} |M_k| > \lambda\} = \frac{1}{\lambda^p} E[|M_n|^p] \qquad (\text{K.I.})$$

$\hookrightarrow$ last index

- In continuous case,

$$P\{\max_{0 \leq s \leq t} |M_s| > \lambda\} \leq \frac{1}{\lambda^p} E[|M_t|^p] \qquad (\text{K.I. in conti.})$$

<u>Define</u>

$$\begin{cases} A_0 = \{w \in \Omega \mid |M_0| > \lambda\} & (\text{get out from initial}) \\ A_1 = \{w \in \Omega \mid |M_1| > \lambda, \ |M_0| \leq \lambda\} & (\text{get out after initial}). \\ A_2 = \{w \in \Omega \mid |M_2| > \lambda, \ |M_1| \leq \lambda, \ |M_0| \leq \lambda\} \\ \vdots \\ A_i = \{w \in \Omega \mid |M_i| > \lambda, \ |M_{i-1}| \leq \lambda, \cdots, |M_0| \leq \lambda\}. \end{cases}$$

clearly. $A_i \cap A_j = \phi \ (i \neq j)$ and $\{w \in \Omega \mid \max_{0 \leq k \leq n} |M_k| > \lambda\} = \bigcup_{k=0}^{n} A_k$

A_k is a set of path that escapes at time k. (exit time $= k$)

$$\Rightarrow P\{\max_{0 \leq k \leq n} |M_k| > \lambda\} = \sum_{k=0}^{n} P(A_k) = \sum_{k=0}^{n} E\{\chi_{A_k}\} \leq \sum_{k=0}^{\infty} E\left[\frac{|M_k|^p}{\lambda^p} \chi_{A_k}\right] \quad (1 \leq p)$$

By Martingale, $|M_k| = |E[M_n | F_k]| \leq E[|M_n| \mid F_k]$

$$\leq E\{|M_n|^p \mid F_k\}^{1/p} \quad (\text{Hölder inequality}).$$

Therefore, $P\{\max_{0 \leq k \leq n} |M_k| > \lambda\} \leq \sum_{k=0}^{n} E\left[E\left\{\left|\frac{M_k}{\lambda}\right|^p \chi_{A_k} \mid F_k\right\}\right]$

$$\leq \sum_{k=0}^{n} E\left[E\left\{\frac{|M_n|^p}{\lambda^p} \mid F_k\right\} \chi_{A_k}\right] =$$

$$= \sum_{k=0}^{n} E\left[E\left[\frac{|M_n|^p}{\lambda^p} \chi_{A_k} \mid F_k\right]\right] = \sum_{k=0}^{n} E\left\{\frac{|M_n|^p}{\lambda^p} \chi_n\right\} = \frac{1}{\lambda^p} E\left[|M_n|^p \sum_{k=0}^{n} \chi_{A_k}\right]$$

Also, note that χ_{A_k} (sum of events) ≤ 1

$\Rightarrow \quad \sim \quad \leq \frac{1}{\lambda^p} \cdot E\{|M_n|^p\}$ #.

$\therefore \quad P\{\max_{0\leq k\leq n} |M_k| > \lambda\} \leq \frac{1}{\lambda^p} \cdot E\{|M_n|^p\}$.

Quadratic variation (QV).

$QV_T(I) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \left(I_{t_{k+1}}(\omega) - I_{t_k}(\omega)\right)^2$ <MSQ> $\max_k (t_{k+1} - t_k) \to 0,\ N \to \infty$

$QV_T(I) = \int_0^T f^2(s,\omega)\,ds \qquad I_T(\omega) = \int_0^T f(s,\omega)\, dB_s(\omega)$

Clearly $\begin{cases} I_T(\omega) = B_T(\omega) & \text{if } f(t,\omega) = 1 \ (0 \leq t \leq T) \\ QV_T(B) = T \end{cases}$ (Every limit → Mean square limit)

Fact (pg 24 in Notes)

Suppose $I(t,\omega) = \int_0^t \sigma(s,\omega)\, dB_s(\omega)$ and $|\sigma(s,\omega)| \leq C \quad 0 \leq s \leq T$
$0 \leq t \leq T$. (Uniformly bounded).

Then, $I^2(t,\omega) - \int_0^t \sigma^2(s,\omega)ds$ is a $\mathcal{F}_t$ Martingale

In fact $I^2(t) - \int_0^t \sigma^2 ds = \int_0^t I(s)\,\sigma(s)\, dB_s$ ~~for this is~~.

By Ito isometry $E\left(\int_0^t I\sigma\, dB_s\right)^2 = E\int_0^t I^2\sigma^2\, ds$

Recall $E\{I^2(t)\} = E\int_0^t \sigma^2(s)\,ds$.

$I^2(t) - \int_0^t \sigma^2 ds \sim$ sq. int.

- Return to Ito's formula,

Suppose $f(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$. f_t, f_x, f_{xx} are bounded.

Then, $f(t, B_t(\omega)) = f(0,0) + \int_0^t f_t(s, B_s(\omega))\,ds + \int_0^t f_x(s, B_s(\omega))\, dB_s(\omega)$

$+ \underbrace{\frac{1}{2}\int_0^t f_{xx}(s, B_s(\omega))\,ds}_{\text{Ito term}}$

If $f = f(x)$, $f(t) = 0$ first term drops out.

Q.V.

$$df(t,B_t) = f_t(t,B_t)\,dt + f_x(t,B_t)\,dB_t + \tfrac{1}{2} f_{xx}(t,B_t)\,d\langle B_t\rangle$$

where $\langle B_t\rangle = QV_t(B) = t.$

$\Rightarrow d\langle B_t\rangle = d\,QV_t(B) = dt.$

so

$$df(t,B_t) = \left\{ f_t(t,B_t) + \tfrac{1}{2} f_{xx}(t,B_t) \right\} dt + f_x(t,B_t)\,dB_t.$$

---

* Recall that $t_0=0,\ t_1,\ t_2,\ \dots,\ t_N=T$ $\quad f(B_t) = f(B_0) + \sum_{k=0}^{N-1} \left(f(B_{t_{k+1}}) - f(B_{t_k})\right).$

(for $f(x)$ with $|f_{xxx}(x)| \le C$) continuous.

$\Rightarrow$ Taylor expansion $\Rightarrow \sum_{k=0}^{N-1} f_x(B_{t_k})(B_{t_{k+1}} - B_{t_k}) + \underbrace{\frac{1}{2}\sum_{k=0}^{N-1} (///)}_{(*)} + \frac{1}{6}\sum_{k=0}^{N-1} \dots$ $\quad f_{xxx}(B_{t_k^*})\Delta B^3 \to 0$ w.p.1

$$\Rightarrow (*) \text{ becomes } = \underbrace{\frac{1}{2}\sum_{k=0}^{N-1} f_{xx}(B_{t_k})\left\{(B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)\right\}}_{\to 0 \ (M.S.Q.)} + \underbrace{\frac{1}{2}\sum_{k=0}^{N-1} f_{xx}(B_{t_k})(t_{k+1} - t_k)}_{= \frac{1}{2}\int_0^T f_{xx}(B_s)\,ds \quad (N\uparrow\infty)}$$

Riemann Integral.

How to show? (sub-division) not the full to $\dots$

$\to$ show it goes to zero, ...

- Full strength Ito's formula

$f(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R},$

$\to f_t(t,x),\ f_x(t,x),\ f_{xx}(t,x)$ bounded, cont.

let $I(t,\omega) = \int_0^t \sigma(s,\omega)\,dB_s(\omega)$, $\quad E\left[\int_0^t \sigma^2(s,\omega)\,ds\right] < \infty$

Then, $$f(T,I(T)) - f(0,I(0)) = \int_0^T f_t(s,I(s))\,ds + \int_0^T f_x(s,I(s))\,dB_s$$
$$+ \frac{1}{2}\int_0^T f_{xx}(s,I(s))\,d\langle I_s\rangle$$

$(dI_s = \sigma(s)dB_s,\ d\langle I_s\rangle = \sigma^2(s)\,ds.)$

Rewrite in diff. form

$$df(t,I(t)) = f_t(t,I(t))\,dt + f_x(t,I(t))\,\underbrace{\sigma(t)\,dB_t}_{(=dI_t)}$$

$$+ \frac{1}{2} f_{xx}(t,I(t))\,\underbrace{\sigma^2(t)\,dt}_{\to d\langle I_t\rangle}$$

(Integrability).

When Ito fails → We need a stopping time

- stopping times
- optimal stopping thm.

01/30/2024.

Stopping theorem (Ito)

$$\boxed{I(t,\omega) = \int_0^t \sigma(s,\omega)\, dB_s(\omega)} \quad \left(\begin{array}{ll} \omega \in \Omega & \Omega = C([0,T];\mathbb{R}) \\ (\Omega, \mathcal{F}_{t, 0\le t\le T}, P) & P = \text{B.M.} \end{array} \right).$$

$\sigma(t,\omega) : [0,T] \times \Omega \to \mathbb{R}$.

$E\int_0^T \sigma^2(t,\cdot)\,dt < \infty$ (non-anticipating).

From Ito's formula,

$$E\left(I(t,\cdot)^2 \right) = E\int_0^t \sigma^2(s,\cdot)\,ds.$$

$$\boxed{\text{Also, } dI(t,\omega) = \sigma(t,\omega)\,dB_t(\omega) \quad (I(0,\omega)=0)}$$

Moreover, $I(t,\omega)$ is continuous and $E[I(t,\cdot) \mid \mathcal{F}_s] = I(s,\cdot)$ (Martingale).

- Let $f(t,x) : [0,T]\times\mathbb{R} \to \mathbb{R}$ with f_t, f_x, f_{xx} bounded.

Let $Y(t,\omega) = f(t, I(t,\omega))$.

then,
$$dY(t,\omega) = f_t(t, I(t,\omega))\,dt + f_x(t, I(t,\omega))\,dI(t,\omega) + \tfrac{1}{2} f_{xx}(t, I(t,\omega))\,\underbrace{d\langle I(t,\omega)\rangle}_{\text{Q.V.}}$$

$$d\langle I(t,\omega)\rangle = \sigma^2(t,\omega)\,dt \quad (\because (*))$$

(*) Note: $QV_t(I) = \langle I_t(t,\omega)\rangle = \int_0^t \sigma^2(s,\omega)\,ds$

pf) Huristic proof.

$$QV_T(I) = \lim_{\substack{N\uparrow\infty \\ \text{in prob.}}} \sup_{\pi_N} \sum_{k=0}^{N-1} \left(I(t_{k+1}) - I(t_k) \right)^2$$

π_N: $t_0 = 0$, ..., $t_N = T$, $\max_k (t_{k+1} - t_k) \to 0$.

$$\sum_{k=0}^{N-1} \left(I(t_{k+1}) - I(t_k) \right)^2 = \sum_{k=0}^{N-1} \left(\int_{t_k}^{t_{k+1}} \sigma(s,\omega)\,dB_s(\omega) \right)^2$$

$$\approx \sum_{k=0}^{N-1} \sigma(t_k,\omega)^2 \left\{ B_{t_{k+1}}(\omega) - B_{t_k}(\omega) \right\}^2 \quad (\because \text{stochastic integral})$$

$$\approx \sum_{k=0}^{N-1} \sigma(t_k,\omega)^2 (t_{k+1} - t_k) \quad (\because \text{QV of BM}).$$

To be more specific,

$$= \sum_{k=0}^{N-1} \sigma^2(t_k, \omega) \cdot \{ (B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k) \} \rightarrow 0 \text{ (goes to zero)}$$

$$+ \sum_{k=0}^{N-1} \sigma^2(t_k, \omega) \, (t_{k+1} - t_k).$$

Ito's formula, $Y(t, \omega) = Y(0, \omega) + \int_0^T f_t(t, I(t,\omega))\, dt$

$$+ \int_0^T f_x(t, I(t,\omega))\, \sigma(t,\omega)\, dB_t(\omega)$$

$$+ \frac{1}{2} \int_0^T f_{xx}(t, I(t,\omega))\, \sigma^2(t,\omega)\, dt.$$

How to prove? $\underline{Y(T,\omega) - Y(0,\omega) = \sum_{k=0}^{N-1} (Y(t_{k+1},\omega) - Y(t_k,\omega))}$

$\rightarrow$ Taylor expansion. DIY.

Example. (application)

Assume. $|\sigma(t,\omega)| \le C$

$E[I^{2p}(t,\cdot)] = E\left[\left(\int_0^t \sigma(s,\omega)\, dB_s(\omega)\right)^{2p}\right]$

(Ito with $f(x) = x^{2p}$) $\quad f_x = 2p\, x^{2p-1} \quad f_{xx} = 2p(2p-1)\, x^{2p-2}$.

$$\Rightarrow I^{2p}(t) = \boxed{\int_0^t 2p\, I^{2p-1}(s)\, \sigma(s)\, dB_s} + \frac{1}{2} \int_0^t 2p(2p-1)\, I^{2p-2}(s)\, \sigma^2(s)\, ds$$

$\downarrow$

we need a guarantee that $E\int_0^t I^{4p-2}(s) < \infty$ to integrate

but, as $p \uparrow$, $4p-2$ is getting too large.

$\Rightarrow$ why don't we argue that $E[\cdot]$ goes to zero

we need stopping time.

Then. $E(I^{2p}(t)) = p(2p-1) \int_0^t E(I^{2p-2}(s))\, \sigma^2(s)\, ds$

$$\le p(2p-1)\, C^2 \int_0^t E[I^{2p-2}]\, ds$$

(if p is real value) Use Hölder inequality

① $p=1$

$E(I^2(t)) \le C^2 t$

② $p=2$

$E(I^4(t)) \le 2 \cdot 3 \cdot C^2 \int_0^t C^2 s\, ds = 3C^4 t^2$

(if Gaussian, $C=1$) $\quad (E(I^{2p}(x)) = 3 \cdot 5 \cdots C^p t^p)$

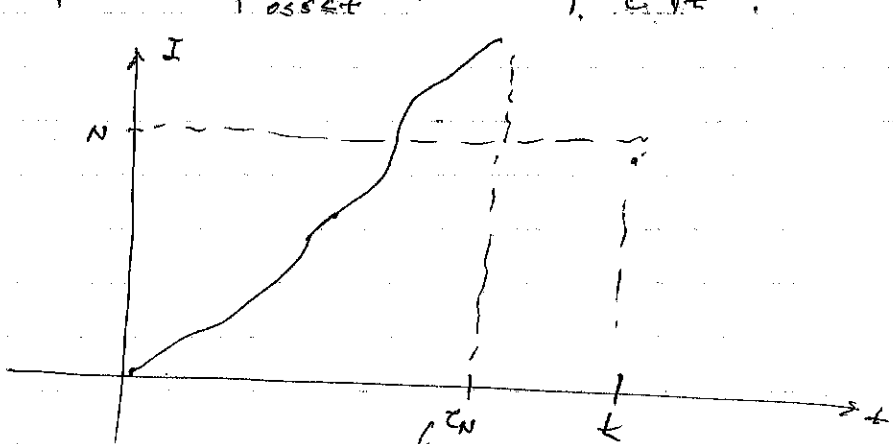
• Stopping time (*) — Simple and Essential

$\tau(\omega) : \Omega \to \mathbb{R}^+$ is a stopping time if for any $t \geq 0$.

$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$.

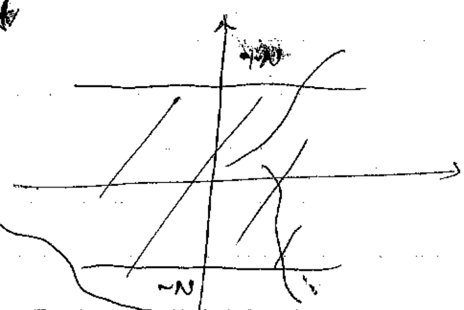
Example.: $\tau_N = \inf\{t \mid I(t;\omega) \geq N\}$ = first time I exceeds 'N'

$\{\tau_N \leq t\} = \{\max_{0 \leq s \leq t} I(s,\omega) \geq N\} \in \mathcal{F}_t$.



∴ τ_N is the first exit time of I from $[-\infty, N]$.

Define $t \times \tau_N = \min(t, \tau_N)$ and $\tau_N = \inf\{t \mid |I(t,\omega)| \geq N\}$



$\Rightarrow \int_0^{t\times\tau_N} 2p\, I^{2p-1}(s)\, \sigma(s)\, dB_s$ —①

$= \int_0^t \chi_{\{\tau_N \leq s\}}\, 2p\, I^{2p-1}(s)\, dB_s$.

Note: optimal stopping theorem

Suppose $M(t)$ is $\mathcal{F}_t$ Mart. $E\{|M(t)|\} < \infty$

and at $0 \leq \rho \leq \tau \leq T < \infty$ ρ, T bounded, stopping times,

then, $E\{M(\tau) \mid \mathcal{F}_\rho\} = M(\rho)$.

$\sigma(s)$ gone!

So, we say ① $= \int_0^{t\times\tau_N} (2p\, I^{2p-1}) \cdot 1\, dB_s$

Also define $I_N(t) = I(t)\ \chi_{\{|N(t)| \le N\}}$ (truncated).

up to time τ_N, $I_N(t) = I(t)$ but after τ_N, $I_N(t) = 0 \ne I(t)$.

Thus, when $|\sigma(t,\omega)| \le C \Rightarrow \tau_N(\omega) \to \infty$.

why? $\Rightarrow P\{\tau_N \le t\} = P\{\max_{0\le s\le t} |I(s)| \ge N\} \le \frac{1}{N^2} E(I^2(t)) \le \frac{C^2 t}{N^2}$ (K.2.)

Stopping times. 02/01/2024.

Recall Ito's formula,

$$\begin{cases} I(t,w) = \int_0^t \sigma(s,w)\, dB_s(w), \quad 0 \le t \le T & \text{———} ① \\ \sigma(t,w) : [0,T] \times \Omega \to \mathbb{R}. & \text{well-defined.} \\ E\int_0^T \sigma^2(s,\cdot)\,ds < \infty \quad (\text{non-anticipating}). \end{cases}$$

$f(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$. , f_t, f_x, f_{xx} bounded.

$$\Rightarrow \boxed{df(t, I(t,w)) = f_t(t, I(t,w))dt + f_x(t, I(t,w))\, dI(t,w) + \frac{1}{2} f_{xx}(t, I(t,w)) \cdot \langle I(t,w)\rangle}$$

where $\langle I(t,w)\rangle = \int_0^t \sigma^2(s,w)\,ds = QV_t(I)$

Let $Y(t,w)$ itself a stochastic integral + deterministic integral.

$$Y(t,w) = Y(0,w) + \int_0^t f_t(s, I(s,w))\,ds + \int_0^t \underbrace{f_x(s, I(s,w))}_{\text{bounded} ✓} \underbrace{\sigma(s,w)}_{\text{sq. integrable.} ✓} dB_s(w)$$

$$+ \frac{1}{2}\int_0^t \underbrace{f_{xx}(s, I(s,w))}_{\text{bounded} ✓} \underbrace{\sigma^2(s,w)}_{\text{sq. integrable}}\,ds \quad ✓$$

Example : How to calculate f (functions) that are not bounded using stopping times.

$\underline{\underline{f(x) = x^{2p}}}$

$\underline{\underline{\text{Calculate}}}$ $E\left(\left(\int_0^t \sigma(s,\cdot)\, dB_s(\cdot)\right)^{2p}\right)$ ($|\sigma(t,w)| \le c$ is needed with suffice.)

⟨Itô⟩

$$dI(t)^{2p} = 2p\, I(t)^{2p-1}\, dI(t) + \frac{1}{2}\cdot 2p(2p-1)\, I(t)^{2p-2}\, \sigma^2(t)\,dt.$$

$$= 2p\, I(t)^{2p-1}\sigma(t)\,dB_t + p(2p-1)\, I^{2p-2}\, \sigma^2(t)\,dt \quad (\because ①)$$

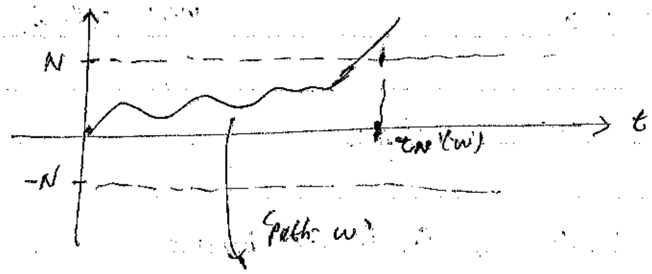
Need $(I^{2p-1})^2 = I^{4p-2}$ is integrable.

$p = 2$ (4th moment)
↓
6th moment of I needed.

• Stopping time comes in...

$\tau(w) : \Omega \to \mathbb{R}^+$, $\{w \in \Omega \mid \tau(w) \le t\} \in \mathcal{F}_t$, $(0 \le t \le T)$

Ex) $\tau_N(w) = \inf\{t \mid |I(t,w)| \ge N\}$



Stopping $\equiv$ Exit (In some sense...)

clearly. $\{\tau(w) \le t\} = \{\max_{0\le s\le t} |I(s)| \ge N\}$: (completely equivalent) $\in \mathcal{F}_t$ (trivial?)

$$P(\tau_N \le t) = P\left(\max_{0\le s\le t} |I(s)| \ge N\right)$$

$$\le \frac{1}{N^2} E\left[\int_0^t \sigma(s,\cdot)\,ds\right] \quad \longrightarrow \text{(K.I)}$$

(stochastic integral are ~~integrable~~ continuous Martingale)

$$\le \frac{c^2 t}{N^2} \quad \text{—— ②}$$

Consider $P\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} (\tau_n \le t)\right) = 0$ implies that

"some" "all" "For some N and any $n \ge N$, $\tau_n \le t$ doesn't happen"

$$= \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty} [\tau_n \le t]\right) \le \lim_{N\to\infty} \sum_{n=N}^{\infty} P(\tau_n \le t) \le \lim_{N\to\infty} c^2 t \sum_{n=N}^{\infty} \frac{1}{n^2} = 0 \quad (\because ②).$$

$\therefore$ $\underline{P\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} (\tau_n \le t)\right) = 0}$. and $\underline{P\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} (\tau_n > t)\right) = 1}$ —— ③

③ means, for some $N_0(w)$ and for all $n \ge N_0(w)$ then, $\tau_n(w) > t$ probability $= 1$.

↳ $\boxed{\text{In short, } \tau_N(w) \to \infty \text{ (w.p.1)}}$

$A \wedge B = \min(A, B)$

- Use stopping time for moments.

$\tau_n = \inf\{t \mid |I(t)| \geq N\}$

$\rightarrow$ Ito's formula up to $\tau_n \wedge t$ is valid. $\because$ (until then I is bounded by N)

Ex)
$$I(t)^{2p} = \int_0^{t\wedge\tau_n} 2p\, I^{2p-1}(s)\, \sigma(s)\, dB_s + \int_0^{t\wedge\tau_n} p(2p-1)\, I^{2p-2}(s)\, \sigma^2(s)\, ds.$$

$$= \int_0^{t\wedge\tau_n} 2p\, I^{2p-1}(s)\, \chi_{\{|I(s)| \leq N\}}\, \sigma(s)\, dB_s \qquad \left(\chi_{I(s)\leq N} = 1 \quad (|I(s)| \leq N)\right)$$
$$+ \int_0^{t\wedge\tau_n} p(2p-1)\, I^{2p-2}(s)\, \sigma^2(s)\, ds.$$

$$M(t) = \int_0^t 2p\, I^{2p-1}(s)\, \chi_{\{|I(s)|\leq N_s\}}\, \sigma(s)\, dB_s. \quad E(M(t)) = 0 \quad \left(\begin{matrix}\text{well-defined}\\ \text{Martingale}\end{matrix}\right).$$

($\tau_n \wedge t \leq t$ is bounded stopping time) $\Rightarrow$ by optional stopping time (OST).

$\Rightarrow E[M(t\wedge\tau_n)] = 0 = M(0)$ also, $E[M(t\wedge\tau_n) \mid \mathcal{F}_0] = M_0 = 0$

$\Rightarrow$ Using this,

$$E[\cancel{I^{2p}(t\wedge\tau_n)}] = E\left[\int_0^{t\wedge\tau_n} p(2p-1)\, I^{2p-2}(s)\, \sigma^2(s)\, ds\right]$$
$$\overset{\|}{I^{2p}(t\wedge\tau_n)}.$$

$$\Rightarrow E[I^{2p}(t\wedge\tau_n)] \leq p(2p-1)\, c^2\, E\left[\int_0^t I^{2p-2}(s)\, ds\right]$$

and also, $\tau_N \to \infty$ w.p.1 as $N \to \infty$

so that, $\tau_N \wedge t \Rightarrow t$ w.p.1 as $N \to \infty$

* $|\sigma(t,\omega)| \leq c$

$(t\wedge\tau_n \uparrow t \quad N \to \infty)$

By Fatu's lemma, (p. 34).

$$E[\lim_{N\to\infty} I^{2p}(t\wedge\tau_N)] \leq \lim_{N\to\infty} E[I^{2p}(t\wedge\tau_N)] \leq p(2p-1)\, c^2\, E\left[\int_0^t I^{2p-2}(s)\, ds\right]$$

$\|$

$E[I^{2p}(t)]$

(w.p.1. as $N \to \infty$).

$p=1)\ E[I^2] \leq c^2 t$

$p=2)\ E[I^4] \leq 2\cdot 3\cdot c^2 \int_0^t c^2 s\, ds = 3c^4t^2$

$\vdots$

$E[I^{2p}] = 3\cdot 5 \cdots (2p-1)\, (c^2 t)^p$

Ex) Exponential Martingale

$$M_\alpha(t) = \exp\left(\alpha I(t) - \frac{\alpha^2}{2}\int_0^t \sigma^2(s)\,ds \right), \quad (\alpha \in \mathbb{R})$$

$M_\alpha(t)$ is an F_t $(0 \le t \le T)$ sq. integrable Martingale. — solution.

and $dM_\alpha(t,\omega) = \alpha\, \sigma(t,\omega) M_\alpha(t,\omega)\, dB_s(\omega)$

<Linear, scalar, Stoch. Diff. Eq.>

To have solution, $\alpha\,\sigma(t,\omega) M_\alpha(t,\omega)$ should be square integrable.

$M_\alpha(0,\omega) = 1$

<Proof of Optional Stopping Theorem> (O.S.T)

Define: $(F_t, 0 \le t \le T)$ Let $\tau \le T$ be a stopping time.

F_τ is the σ-algebra generated by events of the form.

$A \cap \{\tau \le t\} \in F_t$, $A \in F_T$ $(0 \le t \le T)$

$\Rightarrow$ $F_\tau \sim$ Collection of events that depend on path up to time t.

OST Suppose $M(t)$ is a continuous Martingale s.t. $E\{|M(t)|\} < \infty$ $0 \le t \le T$

and if ρ and τ are stop function s.t.

$0 \le \rho(\omega) \le \tau(\omega) \le T < \infty$.

then $\Rightarrow$ $E\{M(\tau) | F_\rho\} = M(\rho)$ (Martingale property is preserved).

Ex) $M(t) = B(t) = B.M$ $B(0) = 0$

let $\tau = \inf\{t \mid B(t) = k\}$

Apply O.S.T,

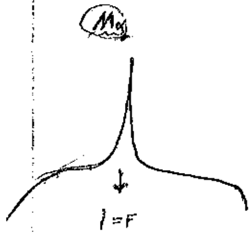
$E\{B(\tau) | F_0\} = B(0) = 0$

O.S.T does not apply, since

$\tau < \infty$ w.p.1 but $E(\tau) = +\infty$

so that there is no $T < \infty$ s.t. $0 \le \tau(\omega) \le T$

HW2 — pb2



02/06/2024.

O.S.T. holds when ρ and τ are bdd. $\Rightarrow E[M_\tau \mid F_\rho] = M_\rho$

(* Martingale property holds for M_τ).

6) $E[M_{t\wedge\tau} \mid F_0] = E[M_{t\wedge\tau}]$? $[a,b]^c$ meaning ?

- Proof of OST → Good practice for conditional expectation (p. 36).
- SDE

$M_\alpha(t) = \exp\left(\alpha I(t) - \alpha^2/2 \int_0^t \sigma^2(s)ds\right)$ $\quad I(t) = \int_0^t \sigma(s)ds$.

$\Rightarrow dM_\alpha(t) = \alpha\,\sigma(t,\omega)\,M_\alpha(t)\,dB_t$ (Ito's)

$\Rightarrow \boxed{M(t,\omega) = 1 + \alpha \int_0^t \sigma(s,\omega)\,M_\alpha(s,\omega)\,dB_s(\omega)}$ → SDE with explicit solution

- SDE

Let $B(t,x)$, $\sigma(t,x)$: $[0,T]\times\mathbb{R} \to \mathbb{R}$. (Ito's condition)

- $|B(t,x)| + \sigma(t,x) \le C(1+|x|)$ for $0\le t\le T$, $x\in\mathbb{R}$; Linear growth cond.
- $|B(t,x) - B(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C(|x-y|)$; Globally Lipschitz cond.

We want to show that the SDE:

$$X_t(\omega) = x + \int_0^t B(s, X_s(\omega))\,ds_t + \int_0^t \sigma(s, X_s(\omega))\,dB_s(\omega)$$

has a unique non-anticipating square integrable solution,

$X_t \in F_t$ and $E[X_t^2] < \infty$ $\quad 0\le t\le T$ is continuous in t.

Thm : Under Ito's condition it is true.

Note : $X_t(\omega) \sim X_t(\omega, x)$ $x\in\mathbb{R}$, $X_t(\omega,x)$ is continuous in x in probability

Comments : ① The case $\sigma \equiv 0$ is included, so all of deterministic ODE is included here.

② The fact that we deal with scalable case is for simplicity only.

③ The linear growth condition is essential in global existence.

Ex) $dX_t = X_t^2\,dt$ $\quad X_0 = 1$ $\Rightarrow$ $X_t = \frac{1}{1-t} \to \infty$ as $t\to 1$

- Lipschitz condition (not for global) is essential for uniqueness.

c.f.) $dX_t = X_t^{2/3}\,dt \quad X_0 = 0$.

we have two solutions, $X_t = 0$, $X_t = t^3/27$.

This is because we don't have <u>Lipschitz property</u>!

★ <u>Grownwell inequality.</u>

If $w(t) \geq 0$ satisfies. $w(t) \leq A + B\int_0^t w(s)ds$.

$A, B > 0$, then $w(t) \leq Ae^{Bt}$.

pf). Define $v(t) = \int_0^t w(s)\,ds$ Then $v(0) = 0$, $dv/dt = w$

so that $dv/dt = A + Bv$ and $dv/dt - Bv \leq A$.

$\Rightarrow e^{Bt}\, d/dt\,(e^{-Bt} v) \leq A$.

$\Rightarrow d/dt\,(e^{-Bt}v) \leq A \Rightarrow e^{-Bt}v(t) \leq A\int_0^t e^{-Bs}ds \Rightarrow v(t) \leq \underline{A/B\,(e^{Bt}-1)}$.

so that, $w(t) \leq A + Bv(t) \leq \underline{Ae^{Bt}}$ #

<u>(1) Uniqueness</u>

Let $X_t^{(1)}$, $X_t^{(2)}$ be two solutions, then,

$$X_t^{(1)} - X_t^{(2)} = \int_0^t \{\underbrace{B(s, X_s^{(1)}) - B(s, X_s^{(2)})}_{\Delta B}\}ds + \int_0^t \underbrace{(\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)}))}_{\Delta\sigma}\, dB_s$$

Use $(a+b)^2 \leq 2a^2 + 2b^2 \Rightarrow (X_t^{(1)} - X_t^{(2)})^2 \leq 2\left(\int_0^t \Delta B\,ds\right)^2 + 2\left(\int_0^t \Delta\sigma\,dB\right)^2$.

$\Rightarrow w(t) = E\left(X_t^{(1)} - X_t^{(2)}\right)^2 \leq 2E\left(\int_0^t \Delta B\,ds\right)^2 + 2E\left(\int_0^t \Delta\sigma\,dB\right)^2$

By global <u>Lipschitz condition</u>, $|\Delta B| \leq C\,|X_s^{(1)} - X_s^{(2)}|$

$|\Delta\sigma| \leq C\,|X_s^{(1)} - X_s^{(2)}|$

$$\Rightarrow w(t) \leq \underbrace{2E\left[t\int_0^t (\Delta B)^2 ds\right]}_{\text{Schwartz ineq.}} + \underbrace{2E\left(\int_0^t (\Delta\sigma)^2 ds\right)}_{\text{Ito isometry}}$$

$$\leq 2(T+1)\int_0^t E[C^2 |X_s^{(1)} - X_s^{(2)}|^2]\,ds$$

$$= 2C^2(T+1)\int_0^t \underbrace{E[|X_s^{(1)} - X_s^{(2)}|^2]}_{w(s)}\,ds$$

$A = 0$

$$w(t) = \frac{2c^2(T-1)}{\beta}\int_0^t w(s)\,ds. \Rightarrow E\left[\left(X_t^{(1)} - X_t^{(2)}\right)^2\right] = 0.$$

$\Rightarrow |X_t^{(1)} - X_t^{(2)}| = 0$ w.p.1 $(0 \le t \le T)$

By continuity, $X_t^{(1)} = X_t^{(2)}$, $0 \le t \le T$ (w.p.1)

since $w(t) \le 2c^2(T+1)\int_0^t w(s)\,ds$. we use Gronwall's inequality.

where $A = 0$, $B = 2c^2(T+1)$.

$\Rightarrow w(t) \le 0 \Rightarrow \boxed{w(t) = 0}$.

$\Rightarrow E\left[|X_t^{(1)} - X_t^{(2)}|^2\right] = 0 \Rightarrow |X_t^{(1)} - X_t^{(2)}| = 0$ w.p.1. $(0 \le t \le T)$

$M_t = \exp(\alpha B_t - \alpha^2 t/2)$ is Martingale.

$E[M_t | F_s] = M_s.$

O.S.T states that,

$E[M(\tau) | F_\rho] = M(\rho)$

for $0 \le \rho(\omega) \le \tau(\omega) \le T < \infty$.

at $\rho = 0$, $E[M_\tau | F_0] = M_0 = 1$.

F_0 contains up to.

$E[M_t | F_0]$

$E[|(M_{t+1} - M_t)| \, | F_t] \le c.$

$= E[\,| \exp(\alpha B_{t+1} - \alpha^2 (t+1)/2) - \exp(\alpha B_t - \alpha^2 t/2)| \, | F_t]$

$= E$

$E[M_{\tau_N}] = E[M_0] = E[1] = 1$

$\|$

$E[\exp(\alpha B_{\tau_N} - \alpha^2 \tau_N / 2)]$

$= E[\exp(\alpha B_{\tau_N})] \, E[-\alpha^2 \tau_N / 2].$

$(\alpha^2/2 = \gamma)$ assume.

$= E[\exp(\sqrt{2\gamma} \cdot B_{\tau_N})]$

$\Rightarrow$ ~~$E[\gamma \tau_N]$~~.

$E[\exp(-\gamma \tau_N)] =$

How to know $E[\exp(\alpha B_{\tau_N})] \cdot \exp(-\alpha^2 \tau_N / 2) = 1$?

$\|$

$\exp(\alpha N)$. why?

$$u(X_{t\wedge\tau}) = u(x) + \int_0^{t\wedge\tau} u'(X_s)\,dX_s + \frac{1}{2}\int_0^{t\wedge\tau} u''(X_s)\,ds.$$

~~exp(∫~~

$$\frac{1}{e^{b-a} - e^{a-b}}\left(e^{x-a} - e^{a-x}\right).$$

pb3. Start from $e^{-rs}u(X_s)$

$c_1 \cosh + c_2 \quad \rightarrow$ use $u(a)=1$, $u(b)=1$

pb4.

$u(X_{t\wedge\tau})$ is Martingale.

Basics — applications of Ito's formula.

- Say we want to evaluate $X(t) = \int_0^t W(s)\,dW(s)$.

By Ito's formula, by taking appropriate $f(x) = x^2/2$.

$$f(W(t)) = f(0) + \int_0^t f'(W(s))\,dW(s) + \frac{1}{2}\int_0^t f''(W(s))\,ds.$$

$$\Rightarrow \quad W(t)^2/2 = 0 + \int_0^t W(s)\,dW(s) \quad \frac{1}{2}\int_0^t 1 \cdot ds$$

$$\therefore \quad X(t) = \int_0^t W(s)\,dW(s) = W(t)^2/2 - t/2$$

Existence of SDE solutions.

$\beta(t,x)$, $\sigma(t,x)$: $[0,T]\times\mathbb{R}\to\mathbb{R}$.

Ito's conditions. (recall)

- $|\beta(t,x)| + |\sigma(t,x)| \le c(1+|x|)$ (growth)
- $|\beta(t,x)-\beta(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le c|x-y|$ (lipschitz) $|x|,|y| \le N$ for any N.

(In time indep. coeffs, the linear growth & local lipschitz $\to$ global lipschitz)

Given β,σ, and $x\in\mathbb{R}$ we want to find $X(t,\omega,x)$ precisely, : $[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$

s.t.

- $X(t,\omega;x)$ is non-anticipating
- $E\{(X(t,\cdot,x))^2\}<\infty$, $0\le t\le T$, $x\in\mathbb{R}$.
- $X(t,\omega;x)$ is continuous, w.p.1

and

$$X(t,\omega,x) = x + \int_0^t \beta(s,X(s,\omega,x))\,ds + \int_0^t \sigma(s,X(s,\omega,x))\,dB_s.$$
$$(0\le t\le T,\ x\in\mathbb{R}\ \text{w.p.1})$$

in short form, $\Rightarrow$ $dX(t) = \beta(t,X(t))\,dt + \sigma(t,X(t))\,dB_t$

$X(0)=x$

(E.g.) $\{\omega\in\Omega \mid X(t,\omega,x)\le y\}\in\mathcal{F}_t$ $\{\Omega=C([0,T]:\mathbb{R}),\ \mathcal{F}_t,\ 0\le t\le T,\ P\}$

↳ True but requires proof.

$\Rightarrow$ we need info. upto time t so that X is (non-anticipating).

(not time over t).

$\to$ we already stated that if we do have a solution then it should be unique

(by using Ito isometry / global lipschitz / gronwall ineq., cont. of X_t, sq. int.)

Next : Square integrability and existence of continuous.

$\to$ Introduce the Peano (1890's) iterates.

$Y_t^{(0)}(\omega) = x$ and for $k=1,2,3,\dots$

$$Y_t^{(k+1)}(\omega) = x + \int_0^t \beta(s, X_s^{(k)}(\omega))\,ds + \int_0^t \sigma(s, X_s^{(k)}(\omega))\,dB_s(\omega).$$

$\to$ continued.

Use $(a+b)^2 \le 2(a^2+b^2)$

$$E[(X_t^{(k+1)})^2] \le 4X^2 + 4E\left[\left(\int_0^t B(s, Y_s^{(k)}) ds\right)^2\right] + 4E\left(\int_0^t \sigma(s, Y_s^{(k)}) dB_s\right)^2$$

$$\le 4X^2 + 4T\, E\int B^2(s, Y_s^{(k)}) ds + 4E\int_0^t \sigma^2(s, Y(s)^{(k)}) ds \quad \text{Ito's isometry}$$

$$B^2 \le C^2(1+|x|)^2$$

$$\le \underbrace{A}_{\text{constant}}(T, x) + B(T)\int_0^t E\{(Y_s^{(k)})^2\} ds.$$

Apply gronwall's inequality,

$$E[(Y_t^{(k)})^2] \le A(T, x)\cdot e^{B(T)t} \quad (0 \le t \le T, \ k = 0, 1, 2, \dots)$$

(uniform in k bound of second moment.)

Consider the difference (to show convergence)

$$Y_t^{(k+1)} - Y_t^{(k)} = \int_0^t \{B(s, Y_s^{(k)}) - B(s, Y_s^{(k-1)})\} ds + \int_0^t (\sigma(s, X_s^{(k)}) - \sigma(s, X_s^{(k-1)})) dB_s$$

let $V^{(k+1)}(t) = E\{(Y_t^{(k+1)} - Y_t^{(k)})^2\}$

$$V^{(k+1)}(t) \le 2C^2(\underset{\substack{B \\ \text{term}}}{T}+\underset{\substack{\sigma \\ \text{term}}}{1})\int_0^t E\{(Y_s^{(k)} - Y_s^{(k-1)})^2\} ds = B\cdot\int_0^t V^{(k)}(s) ds.$$

Start from $k=0$

$$V^{(1)}(t) \le \overbrace{2C^2(1+T)(1+|x|^2)}^{A}\, t$$

$$V^{(2)}(t) \le B\int_0^t As\, ds = BA\, t^2/2 \le (\tilde{A}t)^2/2.$$

$$V^{(k+1)}(t) = E\{Y_t^{(k+1)} - Y_t^{(k)}\} \le (\tilde{A}t)^{k+1}/(k+1)!$$

claim $Y_t^{(k)}(\omega) \to X_t(\omega)$ as $k\to\infty$ uniformly in $0 \le t \le T$ w.p.1 in ω.

$$\Rightarrow Y_t^{(n)}(\omega) = Y_t^{(0)}(\omega) + \underbrace{\sum_{k=0}^{n-1}\left(Y_t^{(k+1)}(\omega) - Y_t^{(k)}(\omega)\right)}_{\text{show this converges uniformly w.p.1}}$$

$$\left|\sum_{k=0}^{n-1}\left(Y_t^{(k+1)} - Y_t^{(k)}\right)\right| \le \underline{\sum_{k=0}^{n-1}\max_{0\le t\le T}\left|Y_t^{(k+1)} - Y_t^{(k)}\right|}$$

let $G^{(k)} = \{\omega\in\Omega \mid \max_{0\le t\le T}|Y_t^{(k+1)} - Y_t^{(k)}| > 1/2^k\}$

Suppose we can show that (proof is in p.44 notes. → use Kolmogorov inequality).

$P(G^{(k)}) \leqq (\tilde{A}T)^{k+1}/(k+1)!$ then by Borrell – Cantelli lemma,

$P(G^{(k)}, \text{i.o.}) = 0 \rightarrow$ means $= \underline{\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} G^{(j)}}$ occurs 0

$\equiv \underbrace{P(\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} G^{(j)})}$ $= 0$

→ This is $\underline{k \geq N(\omega)}$, all $\underline{0\le t\le T}$, $\max_{0\le t\le T}|Y_t^{(k+1)} - Y_t^{(k)}| \le 1/2^k$ w.p.1.

∴ Using this, $\sum(Y_t^{(k+1)} - Y_t^{(k)})$ converges in $0\le t\le T$ w.p.1.

$\Rightarrow Y_t^{(n)}(\omega) - x$ converges to $X_t(\omega)$ ← $\underline{\text{limit of the sum}}$

⇒ By uniform convergence, X_t is cont. in t w.p.1.

by $E[\{Y_t^{(n)}\}^2] \le C$ indep. of t $(0\le t\le T)$.

we concluded that we $\Rightarrow \underline{E\{X_t^2\} \le C}$

Note: $\lim_{n\to\infty} E\{(Y_t^{(n)})^2\} \geq E\{\lim_{n\to\infty}(Y_t^{(n)})^2\}$ (by Fatu's lemma)

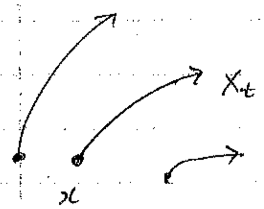
↓ Bounded ⇒ $\underline{(\text{Bounded} \le (1) \le \text{Bounded})}$

idea.

∴ We have the existence & uniqueness of

$$X_t(\omega,x) = x + \int_0^t \beta(s, X_s(\omega,x))\,ds + \int_0^t \sigma(s, X_s(\omega,x))\,dB_s$$

we have shown : $\left\{\begin{array}{l}\text{sq. int.}\\ \text{non-anticipating}\\ \text{cont. int}\end{array}\right.$ ⇒ Show $X_t(\omega,x)$ is cont. in probability in x $\underline{(HW3)}$



$X : \Omega \to \Omega$ for each X

$X : \mathbb{R} \to \mathbb{R}$ for each t

$\hookrightarrow$ This is important concept (dependent on what?)

This is a "Flow"

Next : Markov! $X(t, \omega, x)$ is Markov!

02/13/2024.

What we have done. : 1) Completed SDE (ch5).

- Review SDE basics / briefly. [Basics]

$b(t,x)$ $\sigma(t,x)$: $[0,T] \times \mathbb{R} \to \mathbb{R}$. we have.

Ito condition.

- $|b(t,x)| + |\sigma(t,x)| \le c(1+|x|)$ $\quad (0 \le t \le T \le \infty,\ x \in \mathbb{R})$
- $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C|x-y|$ $\quad (0 \le t \le T,\ x \in \mathbb{R})$

$\to$ There exists a unique $X_t(w,x)$ that (a) non-anticipating / $\boxed{\text{continuous in } t \ \ w.p.1}$

(b) $E\{X_t^2\} \le C$ $\quad 0 \le t \le T,\ C = C(T,x)$

and $X_t(w,x) = x + \int_0^t b(s, X_s(w,x))ds + \int_0^t \sigma(s, X_s(w,s)) dB_s(w)$

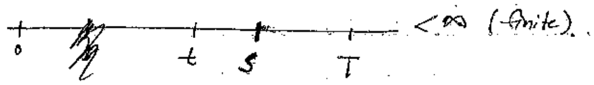
Also, $X_t(w,x)$ is also lipschitz in

$\Rightarrow E\{|X_t(\cdot,x) - X_t(\cdot,y)|\} \le C|x-y|$.

$\therefore$ In fact, $X_t(w,x)$ is continuous w.p.1 in both t and x.

$\hookrightarrow$ Uses Kolmogorov inequality and Burkholder-Gundy inequality.

- Notation



$X_{[s,t]}(w,x)$ is a solution of SDE starting at s from x

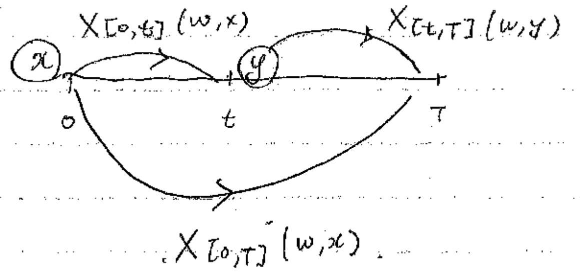
~~$X_{[s,t]}(w,x) = x + \int_s^t b($~~

$X_{[t,T]}(w,x) = x + \int_t^T b(s, X_{[t,s]}(w,x))ds + \int_t^T \sigma(s, X_{[t,s]}(w,x)) dB_s(w)$.

Note that $\{w \in \Omega \mid X_{[t,T]}(w,x) \in A\} \in \mathcal{F}_{[t,T]}$

$\mathcal{F}_{[t,T]} = \sigma\{B_s - B_t \ \ (t \le s \le T)\}$. σ-algebra

$$X_{[0,T]}(\omega, x) = X_{[t,T]}(\omega, X_{[0,t]}(\omega, x)).$$



Geometric interpretation.

$X_{[t,T]}(\omega, y) \in \mathcal{F}_{[t,T]}$ with y fixed

↳ depends on information of $[t,T]$.

key
1) Dynamic composition law of ODE.
2) Stochastic dependence of Brownian path.

Markov property

Suppose $f: \mathbb{R} \to \mathbb{R}$ bdd.

$$E\{f(X_{[0,T]}) \mid \mathcal{F}_t\} = E\{f(X_{[t,T]}(\cdot\, X_{[0,t]}(\cdot\, x))) \mid \mathcal{F}_t\}$$

$(0 \le t \le T < \infty)$.

* composition law

$$= E\{f(X_{[t,T]}(\cdot, X_{[0,t]}(\cdot, x))) \mid X_{[0,t]}\}$$

↳ (only need to know where I landed at time $= t$.)

{That is conditional expectation upto time t is a point function of the path at time t}.

More generally, $f: \mathbb{R}^n \to \mathbb{R}$ and $0 < t < t_1 < t_2 \cdots < t_n < \infty$.

$$E[f(X_{t_n}, X_{t_{n-1}}, \cdots X_{t_1}) \mid \mathcal{F}_t)] = E[f(X_{t_n}, X_{t_{n-1}}, \cdots X_{t_1}) \mid X_t]$$

// ⊥

* They are point functions.

* More notation.

$$E[f(X_{[t,T]}) \mid X_{t_0} = x] = E_{t,x}\{f(X_T)\}. \quad (t<T)$$

Assume. $f(t,x)$, $[0,T] \times \mathbb{R} \to \mathbb{R}$. with f_t, f_x, f_{xx} bdd.

From Ito's formula,

$$\begin{aligned} df(t, X_t) &= f_t\,dt + f_x\,dX_t + \tfrac{1}{2} f_{xx}(dX_t)^2 \\ &= f_t\,dt + f_x \cdot (b\,dt + \sigma\,dB) + \tfrac{1}{2} f_{xx}\,\sigma^2 dt \\ &= \left(f_t + \tfrac{1}{2}\sigma^2 f_{xx} + b f_x\right) dt + \sigma f_x\,dB. \end{aligned}$$

$\Rightarrow$ Integration $\Rightarrow$

$$f(T, X_T) - f(t, X_t) = \int_t^T \left(f_s + \tfrac{1}{2}\sigma^2 f_{xx} + b f_x\right)(s, X_s)\,ds + \int_t^T \sigma f_x(s, X_s)\,dB_s.$$

Let $\mathcal{L}_t = \frac{1}{2}\sigma^2(t,x)\,\partial^2/\partial x^2 + b(t,x)\,\partial/\partial x.$

$$E[f(T, X_T) \mid \mathcal{F}_t] - f(t, X_t) = E\left\{\int_t^T (\partial_s + \mathcal{L}_s) f(s, X_s)\,ds \,\Big|\, \mathcal{F}_t\right\}$$

Suppose $f(t,x)$ is such that $\partial_t f(t,x) + \mathcal{L}_t f(t,x) = 0 \quad t<T$

$f(T,x) = g(T,x)$ is given

$\to$ Backward K.E.

When $\mathcal{L}_t = \frac{1}{2}\,\partial^2/\partial x^2$ B.M. then, $f(t,x) = \int P(T-t, x-y)\,g(T,y)\,dy$

$$P(t,x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$$

Q. Assume that this $f(t,x)$ is such that f_t, f_x, f_{xx} are bounded.

If so, then, $f(t,x) = E\{g(T, X_T) \mid \mathcal{F}_t\} = E_{t,x}\{g(T, X_T)\}$.

Markov property

Two ways.

1) f "solve" the PDE $\Leftrightarrow$ f defined by the expectation solves the PDE.

2) The solution $f(t,x)$ of the PDE can be represented ("solved") probabilistically as an expectation of the solution of SDE.

Indicator function

Let $p(t,x,T,A) = E_{t,x}\{\chi_A(X_T)\}$, $A \subset \mathbb{R}$

$= P_{t,x}\{X_T \in A\}$

which is condition $\mathcal{X}$ starts from x and reaches A.

$\boxed{\text{BKE}}$ is a PDE in $p(t,x,T,A)$ as a function of (t,x).

$$\begin{cases} (\partial/\partial t + \mathcal{L}_t)\, p(t,x,T,A) = 0 & (t<T,\ x \in \mathbb{R}) \\ p(t,x,T,A) = \chi_A(x) & \end{cases}$$

$$\left(\mathcal{L}_t = \frac{1}{2}\sigma^2(t,x)\,\partial^2/\partial x^2 + b(t,x)\,\partial/\partial x \right)$$

If $p(x,t,T,A) = \int \underbrace{p(t,x,T,y)}_{\text{Density}} \chi_A(y)\,dy$

Q) Does $p(t,x,T,y)$, density satisfy a PDE in (T,y)?

Yes, $\rightarrow$ This is $\boxed{\text{FKE}}$

Let $\phi(t,x)$ be a test function. $\phi(0,x) = \phi(T,x) = 0$

Apply Itô to $\phi(t,X_t)$

$$\underset{0}{\underset{\|}{\phi(T,X_T)}} = \underset{0}{\underset{\|}{\phi(t,X_t)}} + \int_t^T (\partial_s + \mathcal{L}_s)\phi\, ds + \int_t^T (\quad)\, ds$$

$$\Rightarrow E_{t,x}\left\{ \int_t^T (\partial_s + \mathcal{L}_s)\phi(s,X_s)\,ds \right\} = 0 = \int dy \int_t^T ds \left(\frac{\partial}{\partial s} + \mathcal{L}_s\right)\phi(s,y)\, p(t,x,s,y)$$

$$= \int dy \int_t^T ds \cdot p(t,x,s,y)\left\{\partial/\partial s + \mathcal{L}_s\right\}\phi(s,y) = 0.$$

Assume $\phi(s,y)$ is differentiable and integrable by parts.

$$0 = \int dy \int_0^T ds\ \phi(s,y)\left[-\partial/\partial s + \mathcal{L}_s^*\right] p(t,x,s,y) \qquad (\text{integrate by parts})$$

$$\Rightarrow \left(-\partial/\partial T + \mathcal{L}_T^*\right) p(t,x,T,y) = 0 \quad \text{for } T>t \quad \text{and} \quad p(t,x,t,y) = \delta(x-y)$$

$$\Rightarrow \mathcal{L}_T^* = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)\,\cdot\right) - \frac{\partial}{\partial y}\left(b(T,y)\,\cdot\right)$$

Therefore, FKE,

$$\partial p/\partial T = \frac{1}{2}\,\partial^2/\partial y^2 \left(\sigma^2 p\right) - \partial/\partial y\,(bp) \quad , (T>t)$$

$$= \partial/\partial y \left\{ \frac{1}{2}\,\partial/\partial y\left(\sigma^2 p\right) - bp \right\}$$

Higher dimensions, $\partial p/\partial T = 1/2\ \nabla\cdot\Big(\underbrace{\cancel{\tfrac{1}{2}}\nabla(\sigma^2 p) - bp}_{\text{Probability flux}}\Big)$ = drift

(Diffusion — Convection Equation)

$$\left(= \frac{1}{2}\nabla\cdot\left\{ \sigma^2\nabla p + 2\left(\nabla\sigma - b\right)p \right\}\right)$$

- Markov property (BKE, FKE). + Girsanov Trans. 02/15/2024.

→ Quick review.

$b(t,x)$ $\sigma(t,x)$: $[0,T]\times\mathbb{R}\to\mathbb{R}$.

< Ito condition >

- $|b(t,x)| + |\sigma(t,x)| \le c(1+|x|)$
- $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le c|x-y|$

$0\le t\le T$ $x\in\mathbb{R}$ there exists a "unique process" $X_t(w), x)$

non-anticipating, continuous t w.p.1 sq. int. $E\{X_t^2\}\le c$ $(0\le t\le T)$ and s.t.

$X_t(w,x) = x + \int_0^t b(s, X_s(w,x))\,ds + \int_0^t \sigma(s, X_s(w,x))\,dB_s(w)$

Define $X_{[t,T]}(w,x)$ process starting from x at time t and going up to time T.

$X_{[t,T]}(w,x) \in \mathcal{F}_{[t,T]} = \sigma\{B_s - B_t,\ t\le s\le T\}$

Moreover, $E\{|X_t(\cdot,x) - X_t(\cdot,y)|^2\} \le C(x-y)^2$

Markov property for any $f:\mathbb{R}^n\to\mathbb{R}$. bdd. and any $(0\le t<t_1<t_2<\cdots<t_n\le T)$

$E\{f(X_{t_n},\cdots,X_{t_1})\,|\,\mathcal{F}_t\} = E\{f(X_{t_n},\cdots X_{t_1})\,|\,X_t\}$

Notation : $p(s,T,x,A) = E\{\chi_A(X_T)\,|\,X_s = x\} = P(X_T\in A\,|\,X_s = x)$ (for $A\subset\mathbb{R}$)

↳ Reaching region A.

I $p(t,x,T,A)$

BKE says that, $p(\underbrace{t,x}_{\text{function of } t,x\text{ s.}},T,A)$

→ Terminal value.

$$\boxed{(\partial/\partial t + \mathcal{L}_t)p = 0\ ,\ t<T \quad p(T,x,T,A) = \chi_A \text{ (indicator)}}$$

where $\mathcal{L}_t\cdot = \frac{1}{2}\sigma^2(t,x)\,\partial^2/\partial x^2 + b(t,x)\,\partial/\partial x$

This is just Itô's formula applied to $p(t, X_t, T, A)$

$\Rightarrow dp = \cancel{(\partial_t + \mathcal{L}_t)p}^{\,0}dt + \sigma(t,X_t)\,\partial/\partial x\ p(t,X_t,T,A)\,dB_t$. (zero because we assume solution of BKE)

$\Rightarrow$ (Integration) $p(T,X_T,T,A) - p(t,x,T,A) = \int_t^T \sigma(s,X_s)\frac{\partial p}{\partial x}(s,X_s,T,A)\,dB_s$.

$\Rightarrow p(t,x,T,A) = E_{t,x}\{\chi_A(X_T)\}$

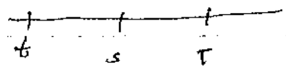
⟨Logic⟩ Assume p solves the BKE → then $p(t,x,T,A) = P_{t,x}(X_T\in A)$

Q) Can we do reverse way? → HW3. Answer : To go backwards, that is to show that $p(t,x,T,A)$ solves the PDE.

Classical way : 1 → Continued.

Q) How to know cond. prob $P(x,t\,;T,A)$ satisfies BKE PDE?

A)



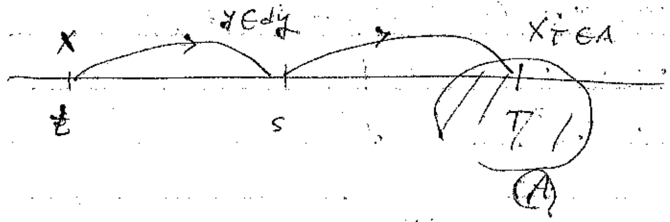
$$P(t,x,T,A) = E\{\chi_A(X_T) \mid X_t = x\}$$

$$= E\{E\{\chi_A(X_T)\mid F_s\} \mid X_t = x\} \quad \text{Iterative Cond. expect.}$$

$$= E\{E\{\chi_A(X_T)\mid X_s\} \mid X_t = x\} \quad \text{Markov prop.}$$

$$= E_{t,x}\{E_{s,X_s}\{\chi_A(X_T)\}\} \quad \text{Notation}$$

$$= \int P(t,x,s,dy)\,P(s,y,T,A) = P(t,x,T,A) \quad \left(\begin{array}{l}\text{Chapman-Kolmogorov}\\ \text{equation}\end{array}\right)$$



$$= \underline{E_{t,x}\{P(s,X_s,T,A)\}} \quad \left(\because P(s,X_s,T,A) = E_{s,X_s}\{\chi_A(X_T)\}\right)$$

Indicator

$\Big\{ P(t,x,T,A) \sim$ family of prob. laws on the real line depending on (t,x,T) satisfying the C-K equation. $\Big\}$

Question: Given $P(t,x,T,A)$ family of laws on the real line satisfy C.K eq.

Does there exist P^X on $(\Omega, F_t, 0\le t\le T)$, $\Omega = C([0,T];\mathbb{R})$

~~s.t. $P(t,x,T,A) = P\{\omega \in \Omega \mid X_T^{(\omega)} \in A,\ X_t^{(\omega)} = x\}$~~

s.t. $P(t,x,T,A) = P^X\{\omega(T)\in A \mid \omega(t) = x\}$

$\Bigg\{$ * Comment: Ito map $(\Omega, F_t, 0\le t\le T, P)$ is the canonical space with $B_t(\omega) = \omega(t)$ being B.M. with P law then by the Ito theory there is a map.

$X_\cdot(\omega;x) : \Omega \to \Omega$

for $0\le t\le T$, $X_t(\omega(\cdot),x)) \in \Omega$.

$\omega(t) \to X_t(\omega)$ < Ito transformation >

$P^X(X_\cdot \in \mathcal{A}) = P\{\omega\in\Omega \mid X_\cdot(\omega)\in\mathcal{A}\}$ $\Bigg\}$

In the question above, we use for a law P^X that transformation $P(t,x,T,A)$

Finite distribution of the X, $\overset{t_0=0}{\vdash}\!\!-\!\!\!-\!\!\!-\!\!\overset{t_N=T}{\dashv}$. $P^X(X_{t_N}\in A_N,\ t_{N-1}\in A_{N-1}\cdots t_0\in A_0)$

For finite dimension distribution,

so, $P^X(X_{t_N} \in A_N, X_{t_{N-1}} \in A_{N-1}, \cdots, X_{t_0} \in A_0)$

$$= \int_{A_0} \cdots \int_{A_N} P_0(dx_0) \left(P(t_0, x_0, t_1, dx_1)\, P(t_1, x_1, t_2, dx_2), \cdots, P(t_{N-1}, x_{N-1}, t_N, dx_N)\right)$$

//

Assume that $P(t, x, T, A)$ satisfies

1) $\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-y| > \delta} P(t, x, t+\Delta t, dy) = 0$. uniform in x and in t. $\forall \delta > 0$

↘ Implies path continuity.

2) $\lim_{\Delta t \to 0} 1/\Delta t \int_{\mathbb{R}} (y-x)\, P(t, x, t+\Delta t, dy) = b(t,x)$.

↘ The $\int$ speed is $b(t,x)$ $\equiv$ Integrated speed is $b(t,x)$.

3) $\lim_{\Delta t \to 0} 1/\Delta t \int_{\mathbb{R}} (y-x)^2\, P(t, x, t+\Delta t, dy) = \sigma^2(t,x)$

↘ Variance of the motion

... Higher moments are not needed (negligible) due to (1).

1), 2), 3) implies that $P(t, x, T, A)$ satisfies $\boxed{\text{BKE}}$

where $\partial P/\partial t + \mathcal{L}_t P = 0$, $t < T$ $x \in \mathbb{R}$

$\mathcal{L}_t = \frac{1}{2}\sigma^2(t,x)\, \partial^2/\partial x^2 + b(t,x)\, \partial/\partial x$

(i) Classical : Derivation from C.K.

(ii) Ito's version : The stochastic calculus implies the BKE. (1, 2, 3)

02/20/2024.

- Girsanov theorem.

Time homogeneous X s.t.

$dX_t = b(X_t)\,dt + \sigma(X_t)\,d\beta_t$, $X_0 = x$.

($b(x), \sigma(x)$ satisfies Ito's condition.)

$$\mathcal{L}_x = \frac{1}{2}\sigma^2(x)\,\partial^2/\partial x^2 + b(x)\,\partial/\partial x$$

let $c(x)$ also satisfies Ito's condition. and assume $\left|\frac{c(x)}{\sigma(x)}\right| \le c$.

so that $M_t = \exp\left(\int_0^t \frac{c(X_s)}{\sigma(X_s)}\,dB_s - \frac{1}{2}\int_0^t\left(\frac{c(X_s)}{\sigma(X_s)}\right)^2 ds\right)$

Q) How to prove? → HW3 pb 2.

and M_t is a F_t non-negative Martingale and by Ito;

$\Rightarrow dM_t = c/\sigma\, M_t\, dB_t + \cancel{\frac{1}{2}(c/\sigma)^2 M_t\, dt} - \cancel{\frac{1}{2}(c/\sigma)^2 M_t\, dt}$

$\therefore M_t = 1 + \int_0^t \underbrace{\frac{c(X_s)}{\sigma(X_s)}}_{\text{bdd} \Rightarrow \text{well-defined}} M_s\, dB_s.$

In particular, $M_t \ge 0$ and $E\{M_t\} = 1$, we can use M_t to change P on Ω, F_t, $0 \le t \le T$.

How?) we can define P^* on Ω, by

$dP^*/dP\big|_{F_t} = M_t$. $\equiv$ If $A \in F_T$, then.

$E^{P^*}\{\chi_A\} = P^*(A) = E^P\{M_T \chi_A\}$. (Note : $P^*(\Omega) = E^P\{M_T \chi_\Omega\} = E^P\{M_T\} = 1$)

Moreover, if $A \in F_t$, then $P^*(A) = E^P\{M_T \chi_A\} = E^P\{E^P[M_T \chi_A | F_t]\}$

$= E^P\{\chi_A \underbrace{E^P(M_T | F_t)}_{= M_t}\} = E^P\{\chi_A M_t\}$

$\therefore dP^*/dP\big|_{F_t} = M_t$

Note that if $P(A) = 0$, $A \in F_t$, Then $P^*(A) = 0$.

$P^*(A) = 0$ then $P(A) = 0$ → P^* and P are equivalent

But we need Girsanov theorem, $\equiv$ shares 0 probability.

we $D = $

Girsanov thm.

$$\boxed{\begin{array}{l} X \text{ under } P^* \text{ has same law as } Y \text{ under } P, \text{ where} \\ dY_t = (b(Y_t) + c(Y_t))dt + \sigma(Y_t)\, d\tilde{B}_t \\ \qquad (\text{where } \tilde{B}_t \text{ is another B.M.}) \end{array}}$$

$B_t(w) = w(t) \Rightarrow \tilde{B}_t(w) = \int_0^t \sigma_s(w)\, dB_s(w)$

where $\sigma_t(w)$ is non-anticipating and $|\sigma(t,w)| = 1$ $0 < t \leq T$

Follows by levy characterization since $\langle \tilde{B}_t \rangle = \int_0^t |\sigma_s|^2 ds = t$

$\Rightarrow$ $\underline{X \text{ under } P^* \sim Y \text{ under } P}$ (Girsanov)

Essential calculation

$$dX_t = b(X_t)dt + \sigma(X_t)\, dB_t = (b(X_t) + c(X_t))dt + \left(-\frac{c(X_t)}{\sigma(X_t)}dt + dB_t\right)\sigma(X_t)$$

(addition and subtraction)

For Girsanov to hold, we must have $\underline{-\frac{c(X_t)}{\sigma(X_t)}dt + dB_t = dB_t^*}$

is a B.M under P^*

$B_t \to B_t^*$ $\boxed{\text{* Brownian in } P \text{ with drift is also a Brownian in different } P^*}$

$\Rightarrow$ Matter of perspective $\underline{(P \text{ and } P^*)}$

Variant — Importance sampling

consider X under P and change $X \to Y$ by adding (replacing) b by $b+c$.
as well as P by P^* where $\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = M_t$ with $M_t = \exp\left(-\int_0^t \frac{c(Y_s)}{\sigma(Y_s)} dB_s - \frac{1}{2}\int_0^t \left(\frac{c(Y_s)}{\sigma(Y_s)}\right)^2 ds\right)$.

Then, Y under P^* is the same as X under P

↳ Brownian bridge.

Pf) $\cdot\, d(f(t, X_t)\cdot M_t) = M_t\, df + f\,dM + df\,dM$

$$= M\left((f_t + f_x f)\,dt + \sigma f_x\,dB\right) + f\left(\frac{c}{\sigma} M\,dB\right) + \sigma f_x \frac{c}{\sigma} M\,dt$$

$$= M_t\,(f_t + \underbrace{f_x f_t + c f_x}_{\mathcal{L}_x f})\,dt + \left(f\,\frac{c}{\sigma} + \sigma\,dx\right) M\,dB.$$

$$= M_t\,(f_t + \mathcal{L}_x f)\,dt + \left(f\,\frac{c}{\sigma} + \sigma f_x\right) M_t \cdot dB_t.$$

$$E\left\{ fM\Big|_t^T \;\Big|\; \mathcal{F}_t\right\} = E\left\{\int_t^T M_t\,(f_t + \mathcal{L}_y f)\,dt \;\Big|\; \mathcal{F}_t\right\}.$$

Suppose $\begin{cases} f_t + \mathcal{L}_y f = 0 & (t < T) \\ f(T, y) = g(y) \end{cases}$ (BKE)

Then $f(t, X_t)\,M_t = E_{t,x}\{g(Y_T)\,M_T\}$

$$\Rightarrow f(t,x) = E_{t,x}\{M_T/M_t \cdot g(Y_T)\} = \underline{\underline{E^{P^*}_{t,x}(g(Y_T))}}$$

∴ Y_t under P^* has the law associated with SDE,

$$\underline{dY(t) = (bY_t + cY_t)\,dt + \sigma(Y_t)\,d\hat{B}_t}$$

Used implicitly that the BKE uniquely determines the law of probability.

X paths — set of paths

$$P(X_\cdot \in A) = P(w \in \Omega \mid X_\cdot(w) \in A) \equiv P^X(A)$$

P^X is a new prob law on $(\Omega, F_t, 0 \le t \le T)$

$\Rightarrow X \sim P$ means the law P^X

$$dX = b\,dt + \sigma\,dB = (b+c)\,dt + \sigma\underbrace{\left(-\tfrac{c}{\sigma}\,dt + dB\right)}_{d\hat{B}^*}$$

B^* is B.M under P^* $\Leftrightarrow$ $Y \sim P^*$ is the law of $dY = (b+c)\,dt + \sigma\,d\tilde{B}$

* Martingale representation theorem *

Girsanov theorem continued. 02/22/2024.

Show that $B_t^* = B_t - \int_0^t \frac{c(X_s)}{\sigma(X_s)} ds$ is B.M. under P^*

$E^{P^*}\{(B_t^*)^2 - t \mid F_s\} = (B_s^*)^2 - s$ (show this)

$\Rightarrow E^{P^*}\{(B_t^{*2} - B_s^{*2}) \mid F_s\} = t - s$

$\Rightarrow E^{P^*}\{(B_t^* - B_s^*)^2 - 2B_s^{*2} + 2B_t^* B_s^* \mid F_s\}$

$= E^{P^*}\{(B_t^* - B_s^*)^2 \mid F_s\} = \underline{E^P\left[\frac{M_t}{M_s} \cdot \left\{(B_t - B_s) - \int_s^t \frac{c}{\sigma} ds'\right]\right.}$

$\Rightarrow$ Use Itô's formula, $M_{\alpha,t} = \exp\{\alpha B_t^* - \frac{\alpha^2}{2} t\}$

show $N_{\alpha,t}$ is a P^* is a martingale

$d(N_\alpha M) = N_\alpha dN + M dN_\alpha + dN_\alpha dM = N_\alpha\left(\frac{c}{\sigma} M\right) dB$
$+ M(\alpha N_\alpha dB^* - \frac{\alpha^2}{2} N_\alpha dt)$
$+ dN_\alpha dM.$

$\underline{= N_\alpha \frac{c}{\sigma} M dB + M(\alpha N_\alpha (dB - \frac{c}{\sigma} dt) + \frac{\alpha^2}{2} N_\alpha dt)}.$

$dM = c/\sigma\, M dB.$

$dN_\alpha = d e^{\alpha(B_t - \int_0^t c/\sigma\, ds) - \frac{\alpha^2}{2} t} = \alpha N_\alpha dB_t + \cancel{\frac{1}{2}\alpha^2 N_\alpha dt} - \frac{\alpha c}{\sigma} N_\alpha dt - \cancel{\frac{\alpha^2}{2} N_\alpha dt}$

$= \frac{c}{\sigma} M dB N_\alpha + M(\alpha N_\alpha dB_t - \cancel{\frac{\alpha c}{\sigma} N_\alpha dt}) + \cancel{\frac{c}{\sigma} M_\alpha N_\alpha dt}$

$\underline{= c/\sigma\, M N_\alpha dB + \alpha M N_\alpha dB}.$

Messy calculations

$= M N_\alpha (\alpha + c/\sigma) dB$ = martingale.

$\Rightarrow \underline{E^{P^*}\{N_{\alpha,T} \mid F_t\} = N_{\alpha,t}}.$ $\alpha \in \mathbb{R}$

• Brownian bridge,

Define a new P^* $(0 < s \le t \le T)$

$$dP^*/dP = \frac{u(t, X_t; T)}{u(s, x; T)} \rightsquigarrow M_t$$

$$d \log(u(t, X_t)) = \frac{1}{u} u_x \, dX_t + \frac{1}{u} u_t \, dt - \frac{1}{2}\left(\frac{1}{u^2} u_x^2\right) + \frac{1}{2}\left(\left(\frac{1}{u} u_x\right)' (dX_t)^2\right)$$

$$= \frac{1}{u} u_x \, dX_t + \frac{1}{u} u_t \, dt - \cdots$$

$$= \frac{1}{u} u_x (b\,dt + \sigma\, dB) + u_t/u \, dt + 1/2\left(\frac{u_{xx}}{u} - u_x^2/u^2\right)\sigma^2 dt.$$

$$\Rightarrow d \log(u(t, X_t)) = (u_x/u \; \sigma)\, dB + \left(-\frac{1}{2}\left(\frac{\sigma u_x}{u}\right)^2 dt\right),$$

$$\log\left(\frac{u(t, X_t)}{u(s, X_s)}\right) = \int_s^t (?)$$

$$=$$

By Girsanov,

X under P^* is like Y under $\tilde{P}$. where Y is. $dY = (b + u_x \sigma^2/u)\, dt + d\tilde{B}$

$B \cdot B$ $b = 0$, $\sigma = 1$, $dX = dB$. (B.M)

$$\Rightarrow u(t, x, T) = \underbrace{\exp(-x/2(T-t))/\sqrt{2\pi(T-t)}}_{\prime\prime} \Rightarrow u_x/u = -\frac{x-z}{T-t}$$

Y spc

$$\Rightarrow dY_t = -\frac{(Y_t - z)}{T-t} dt + d\tilde{B}, \quad Y_s = x,$$

Singular O.U. process.

$Y_t \quad s \le t \le T \rightarrow$ $Y_T = z$

나의 단점 :
1. $\underline{\text{질문하기}}$ (모르면 ...)
2. paper 잘 쓰기
3. MD 잘 쓰기

$E^{P^*}\{\underset{\sim}{\cdot}\} = E^{P}\{M_t \underset{\sim}{\cdot}\}$ where $M_t = \exp(\sim)$ in the notes.

$$k(i \to i+1) = \exp\left(-\frac{E(i+1) - E(i)}{2k_BT}\right)$$

$$I(i+1)/I(i) = \exp\left(-\frac{E_b(i+1) - E_b(i)}{2k_BT}\right) \to k'(i \to i+1) = \exp\left(-\frac{E_{Tot}(i+1) - E_{Tot}(i)}{2k_BT}\right)$$

$$dY_t = \underline{Y_t - z}$$

E.g. O.U. process

$$dX_t = -\alpha X_t\,dt + \sigma\,dB_t \qquad X_0 = x$$

$$\underbrace{dX_t + \alpha X_t\,dt}_{} = \sigma\,dB_t$$

$$\Rightarrow \underbrace{e^{-\alpha t}\,d(e^{\alpha t} X_t)}_{\text{Integ. factor}} = \sigma\,dB_t$$

$$\Rightarrow e^{\alpha t} X_t \Big|_t^T = \sigma \int_t^T e^{\alpha s}\,dB_s$$

$$\Rightarrow \cancel{\quad} \int_t^T$$

$$e^{\alpha T} X_T - e^{\alpha t} X_t = \sigma \int_t^T e^{\alpha s}\,dB_s$$

$$\therefore X_T = \underbrace{e^{-\alpha(T-t)} X_t}_{\text{mean}} + \sigma \int_t^T e^{-\alpha(T-s)}\,dB_s$$

$$\cancel{X_T \sim N(e^{-\alpha(T-t)} \cdots}$$

$$X_T \sim N(e^{-\alpha(T-t)}x$$

$$\mathrm{Var}(X_T) = \sigma^2 E\left\{\left(\int_t^T e^{-\alpha(T-s)}\,dB_s\right)^2\right\}$$

$$= \sigma^2 \int_t^T e^{-2\alpha(T-s)}\,ds \quad \langle \underline{\text{Ito - isometry}} \rangle$$

$$= \underline{\frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)}$$

$$\shortparallel$$

$$\underline{\mathrm{Var}} = (\sigma_{X_T})^2$$

Brownian bridge.

$$dX_t = -\frac{X_t - z}{T-t}\,dt + \sigma\,dB_t$$

$X_t - z \to X_t$

$$\Rightarrow \quad dX_t = -\frac{X_t}{T-t}\,dt + \sigma\,dB_t$$

$$\Rightarrow \left(dX_t + \frac{X_t}{T-t}\,dt \right) = \sigma\,dB_t$$

$\downarrow$ Int. Fac.

$$(T-t)\,d\left(\frac{1}{T-t}\,X_t\right) = \sigma\,dB_t$$

$$\left(\frac{1}{T-t}\,X_t\right)\Big|_s^t = \sigma\int_s^t \left(\frac{1}{T-s'}\right) dB_{s'}$$

$$\Rightarrow \quad X_t = \frac{T-t}{T-s}\,X_s + \left(\sigma\int_s^t \frac{1}{T-s'}\,dB_{s'}\right)(T-t)$$

$$\Rightarrow \quad X_t = z + \frac{T-t}{T-s}\,X + \sigma\,(T-t)\int_s^t \frac{1}{T-s'}\,dB_{s'}$$

$$\begin{pmatrix} X_s = X \\ X_t = z \end{pmatrix}$$

- Maximum likelihood.

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dB_t \qquad Y_0 = x$$

SDE

$$dX_t = \sigma(X_t)\,dB_t \qquad X_0 = x$$

In discrete time,

$$\begin{cases} Y_{n+1} = Y_n + b(Y_n)\cdot\Delta t + \sigma(Y_n)\,\Delta B_{n+1} \\ X_{n+1} = X_n + \sigma(X_n)\,\Delta B_{n+1} \end{cases}$$

S diff Eq.

→ one step forward = Euler

$$\Delta B_{n+1} = B_{(n+1)\Delta t} - B_{n\Delta t}$$

<u>Approximation thm.</u> $P\left(\max_{0\le k\le n} |Y_{k\Delta t} - Y_k| > \delta\right) \to 0.$

$Y_{k\Delta t}$: From SDE; Y_k: From S diff. Eq

for all $\delta > 0$ as $\Delta t \to 0$, $n\Delta t = T$, $n \to \infty$

This is called Euler expression of <u>SDE</u>.

From B.M.

$$P^*(Y_{k+1} \mid Y_k) = N\left((Y_k + b(Y_k)\Delta t),\ \underline{\sigma^2(Y_k)\,\Delta t}\right)$$

$$= \exp\left(-\frac{(Y_{k+1} - Y_k - b(Y_k)\Delta t)^2}{2\sigma^2(Y_k)\,\Delta t}\right) \Big/ \sqrt{2\pi\sigma^2(Y_k)\,\Delta t}$$

How about path?

$$P^*(Y_n, Y_{n-1}, \cdots, Y_1 \mid Y_0) \overset{\text{Markov prop}}{=} \prod_{k=0}^{n-1} P^*(Y_{k+1} \mid Y_k)$$

$$M_n(X_n, \cdots X_1, X_0) = \frac{P^*(X_{n+1}, X_n, \cdots, X_1, X_0)}{P(X_{n+1}, \cdots, X_1, X_0)} = \prod_{k=0}^{n-1} \frac{P^*(X_{k+1}, X_k)}{P(X_{k+1}, X_k)}$$

$$\Rightarrow E^P(M_n \mid X_0, \cdots, X_{n-1}) = M_{n-1}$$

So M_n is P martingale.

$$E^P\left[\prod_{k=0}^{n-1} \frac{p^*(X_{k+1}|X_k)}{p(X_{k+1}|X_k)} \Bigg| X_0, \cdots, X_{n-1} \right]$$

$$= \prod_{k=0}^{n-2} \frac{p^*(X_{k+1}|X_k)}{p(X_{k+1}|X_k)} \int \frac{p^*(X_n|X_{n-1})}{\cancel{p(X_n|X_{n-1})}} \cdot \cancel{p(X_n|X_{n-1})}\, dX_n$$

$$= M_{n-1}(X_0 \sim X_{n-1})$$

Summary $E^P(M_n) = 1$

$$\underline{E^{P^*}(f(X_n, \cdots, X_0)) = E^P(f(X_n, \cdots, X_0)\, M_n)} \quad //$$

Recall what M_n is,

$$M_n(X_n, X_{n-1}, \cdots, X_0) = \prod_{k=0}^{n-1} \frac{p^*(X_{k+1}|X_k)}{p(X_{k+1}|X_k)}$$

$$= \prod_{k=0}^{n-1} \exp\left(- \frac{(X_{k+1} - X_k - b(X_k)\Delta t)^2 - (X_{k+1} - X_k)^2}{2\sigma^2(X_n)\Delta t} \right)$$

$$= \prod_{k=0}^{n-1} \exp\left(- \frac{1}{2\sigma^2(X_k)\Delta t} \left[\, (??) \, \right] \right)$$

$$= \prod_{k=0}^{n-1} \exp\left(+ \frac{(X_{k+1} - X_k)\, b(X_k)}{\sigma^2(X_k)} - \frac{1}{2}\left(\frac{b(X_k)}{\sigma(X_k)} \right)^2 \Delta t \right)$$

$$\downarrow$$

but $X_{k+1} - X_k = \sigma(X_k)\, \Delta B_{k+1}$

$$\downarrow$$

(p. 84 ~ 85)

Max. Like. Thm. 02/29/2024.

$p^*(Y_{k+1} | Y_k) \rightarrow$ Likelihood (observation)

$\underbrace{p^*(Y_N \cdots Y_0)}_{\text{prob}} = \prod_{i=0} p^*(Y_{i+1} | Y_i)$

$$M_n = \frac{p^*(X_{k+1} | X_k) \cdots}{p(X_{k+1} | X_k) \cdots} = \exp\Big(\underbrace{A (X_{k+1} - X_k)}_{\text{Stoch. int.}} + \underbrace{B \Delta t}_{\text{Riemn. int.}} \Big)$$

↳ square terms are gone!

$M_n(x, b, \sigma)$; observe X. unknown b & σ.

Good estimate of $(b, \sigma) \rightarrow$ maximizes M_n → This is Martingale.

↳ * This estimator is $\boxed{\text{inconsistent}}$ and gets accurate as $t \rightarrow \infty$ $(n \rightarrow \infty)$

- Mean reversion → asset price will converge to its average

Evolved.

- Monte Carlo. 03/05/2024.

X, Y uniform and $\ln(X^2+Y^2) = R$

$\Theta = \tan^{-1}(X/\sqrt{X^2+Y^2})$

Then, ($R\cos\theta$ and $R\sin\theta$) are Gaussian.

$u^N(t,x) = \frac{1}{N}\sum_{n=1}^{N} g(X_t^{(n)})$ (X_t is solution of SDE)

$E\{u^N(t,x)\} = E\{\frac{1}{N}\sum g(X_t^{(n)})\} = E\{g(X_t)\} = \underline{u(t,x)}$

$\therefore$ $u^N(t,x)$ is unbiased respect to $u(t,x)$.

$$\Rightarrow Var(u^N) = E\left\{\left(\frac{1}{N}\sum_{n=1}^{N}\left(g(X_t^{(n)}) - u(t,x)\right)\right)^2\right\}$$

$$= \frac{1}{N^2}\left[\sum_{n=1}^{N} E\{g(X_t^{(n)} - u(t,x))^2\}\right]$$

$$= \frac{1}{N}\cdot E\{(g(X_t) - u(t,x))^2\} = \underline{O(1/N)}$$

$\therefore$ Std (σ) goes down like $1/\sqrt{N}$ $\rightsquigarrow$ accuracy

Question 1) Calculate $E\{(g(X_t) - u(t,x))^2\}$ by Monte-Carlo.

" 2) Can this factor reduced.

Let $V(t,x)$ solve $\begin{cases} V_t = \mathcal{L}V, & t>0 \\ V(0,x) = g^2(x) \end{cases}$ Recall that $\begin{cases} u_t = \mathcal{L}u & t>0 \\ u(0,x) = g(x) \end{cases}$

Apply Ito's formula,

$$d\{V(t-s, X_s) - u^2(t-s, X_s)\} = [(-V_s + \mathcal{L}V)ds + \sigma V_x dB]$$
$$- [2u u_s ds + 2u u_x dX + \underbrace{(u u_x)_x}_{= u u_{xx} + u_x^2}(dX)^2]$$

$$\Rightarrow \sigma V_x dB - 2u u_x \sigma dB + [-2u(-u_s + \mathcal{L}u)ds + u_x^2\sigma^2 ds]$$

$$= -\sigma^2 u_x^2 ds - (2u u_x\sigma - \sigma V_x) dB$$

Integrate $s \to [0,t]$

Integrate $s \to [0,t]$

$$E_x \{ v(t-s, X_s) - u^2(t-s, X_s) \}_0^t = -E_x \int_0^t \sigma v_x^2 \, ds.$$

$$\Rightarrow E_x \{ -(v(t,x) - u^2(t,x) \} = -E_x \int_0^t \sigma^2 u_x^2 \, ds$$

$$N \operatorname{Var}(u^N) = v(t,x) - u^2(t,x) = E_x \int_0^t \sigma^2(X_s) \, u^2(t-s, X_s) \, ds.$$

- why do this? → Error can be computed recursively

- Cannot do $u_x(t,x) \xrightarrow{X} (u^N(t,x))_N$

↳ you $\boxed{\text{should not differentiate}}$ in numerical sim

Good way → $X_t = X(t,x) = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s.$

Assuming regularity of coefficients, (b, σ).

$$Z_t = \frac{\partial}{\partial x} X(t,x), \quad Z_t = 1 + \int_0^t b_x(X_s) \underbrace{Z_s}_{\text{Chain rule}} ds + \int_0^t \sigma_x(X_s) Z_s \, dB_s.$$

Jointly (X_t, Z_t) are a Markov process.

$$\Rightarrow d\begin{pmatrix} X_t \\ Z_t \end{pmatrix} = \begin{pmatrix} b(X_t) \\ b_x(X_t) Z_t \end{pmatrix} dt + \begin{pmatrix} \sigma(X_t) \\ \sigma_x(X_t) Z_t \end{pmatrix} dB_t \qquad \left(\begin{pmatrix} X_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix} \right)$$

M.C. → generate jointly $X_t^{(n)}$ $Z_t^{(n)}$

Then, $\begin{cases} u^N(t,x) = \frac{1}{N} \sum g(X_t^{(n)}) \\ u_x^N(t,x) = \frac{1}{N} \sum g_x(X_t^{(n)}) \, Z_t^{(n)} \end{cases}$

$$N \operatorname{var}(u^N) \sim \frac{1}{N} \sum \int_0^t \sigma^2(X_s^{(n)}) \, u_x^N(t-s, X_s^{(n)}) \, ds.$$

Variance reduction. 03/05/2024.

- $dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$. $(X_0 = x)$

Introduce operator $\mathcal{L}_c = \frac{1}{2}\sigma^2(x)\,\partial^2/\partial x^2 + \{b(x) + c(x)\}\,\partial/\partial x$ $\left(\left|\frac{c(x)}{\sigma(x)}\right| \le c\right)$

Let $M_{c,t} = \exp\left(-\int_0^t \frac{c(X_s)}{\sigma(X_s)}\,dB_s - \frac{1}{2}\int_0^t \left(\frac{c(X_s)}{\sigma(X_s)}\right)^2 ds\right)$

where $dY_t = \{b(Y_t) + c(Y_t)\}\,dt + \sigma(Y_t)\,dB_t$

By Girsanov, (Variant),

law $(X_t, \mathcal{L}_0, P)$ is same as $(Y_t, \mathcal{L}_c, P^*)$ where $dP^*/dP\big|_{\mathcal{F}_t} = M_{c,t}$

So $u(t,x) = E_x^P\{g(X_t)\} = E_x^{P^*}\{g(Y_t)\}$

$= \underline{E_x^P\{g(Y_t)\,M_{c,t}\}}$

Now, $c = c(t,x)$ to be chosen to reduce the variance of $u^N(t,x)$

But, $u^N(t,x) = \frac{1}{N}\sum_{n=1}^{N} g(Y_t^{(n)})\,M_{c,t}^{(n)}$

n^{th} indep. path of process Y

Clearly $\underset{G.T.}{E\{u^N(t,x)\} =} u(t,x)$ (unbiased)

What about $\operatorname{var}(u^N)$?

$$N \cdot \operatorname{var}(u^N) = E_x^P\left\{\int_0^t \left[(\sigma(Y_s)\,u_x(t-s, Y_s))^2 - \left(u(t-s, Y_s)\,\sigma(t-s, Y_s)/\sigma(Y_s)\right)\right] M_{c,s}^2\,ds\right\}$$

$\boxed{\text{Best } c}$ → Makes variance zero

$$c(t,y) = \frac{\sigma^2(y)\cdot u_x(t,y)}{u(t,y)} = \sigma^2(y)\,\ln(u(t,y))\big|_y$$

Ch.10. Review. 03/07/2024

$u(t,x) = E_x[g(X_t)]$ solves B.K.E (PDE)

→ Run M.C. simulation to get $E[\,]$ value.

$b(X_t)$ $\sigma(X_t)$ are not known exactly (e.g. $\sigma \sim N(0, \Delta t)$)

→ Use M.C. simulation (accuracy is $O(1/n)$ very low)

→ probablistic & large dimension → M.C. simulation.

$u_t = \mathcal{L}u.$ $u|_{t=0} = g$ (start from here)

start from $u_{(0)}$

↪ calculate $\sigma^2 u_x / u = c(0)$

↪ $u_{(1)} \to c_{(1)}$

↪ ... (repeat)

$c^{(m)}$ reduces the variance of M.C. estimates.

$q(x)$ satisfies BKE

③ $X, Y = re^{-(\alpha - 1/2)}(\cos \beta t, \sin \beta t)$

HW4

① $X, Y = (R \cos z_t, R \sin z_t)$

Ito's → z_t is BM

$dX = -R \sin z\, dz - \frac{1}{2} r \cos z\, dt$ (Ito)

$= -\frac{1}{2} Y \|dz\|^2 + (-Y dz)$

② $\tan^{-1}(Y/X) = \theta$

$dY = -\frac{1}{2} X \|dz\|^2 + X dz_t$

$z_t = \tan^{-1}(Y_t/X_t)$

① $\to (X, P, \mathcal{L}) \xrightarrow[b \to b+c]{c} (Y, \cdot, \mathcal{L}_c)$

② $\to$ $M_c = \int_0^t$

③ $dP^*/dP = M_c$

$$dY = (b+c)\,dt + \sigma d\tilde{B}_t$$

$\to U$