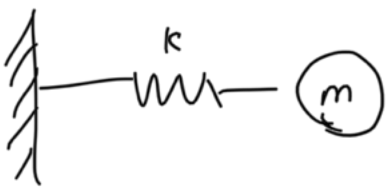
Wave Lecture 1 (20210831)

Lec 1



① Newtonian mechanics. $F = ma \Rightarrow m\ddot{x} + kx = 0.$

② Lagrangian mechanics based on Hamilton principle

Hamilton principle $\Rightarrow$ [The path of a particle, $x(t)$, is determined such that the function $I(x, \dot{x}) = \int_{t_1}^{t_2} L(t, x, \dot{x})\, dt$ has an extremum]

For I to have extremum, $(L = T - V)$.

$$\boxed{\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0}$$: Euler - Lagrange equation

$L = T - V$; $T = \frac{1}{2} m (\dot{x})^2$, $V = \frac{1}{2} k x^2$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$m\ddot{x} + kx = 0$$

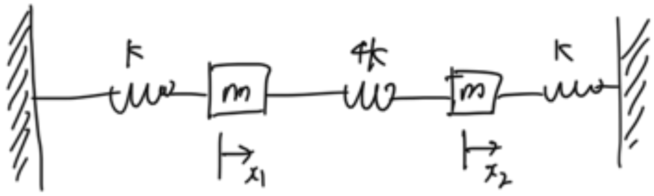
$-\nabla V = $ force.

$\Rightarrow -\frac{dV}{dx} = -kx.$

$\Rightarrow V = \frac{1}{2} kx^2 + C.$

Wave Lecture 2 (20190902)

$\boxed{\text{Lec 2}}$ < Review of "Vibration" >



Using Lagrangian Mechanics, $L = T - V$

$$\begin{cases} T = \frac{1}{2}m(\dot{x}_1)^2 + \frac{1}{2}m(\dot{x}_2)^2 \\ V = \frac{1}{2}k x_1^2 + \frac{1}{2}k x_2^2 + \frac{1}{2}4k(x_1 - x_2)^2 \end{cases}$$

$\Rightarrow$ E.O.M by Lagrangian Mechanics. (Euler - Lagrange Eq.)

$$\left.\partial L/\partial x_i - \partial/\partial t\left(\partial L/\partial \dot{x}_i\right) = 0\right|_{i=1,2} \Rightarrow \left.\begin{array}{l} -kx_1 - 4k(x_1 - x_2) - \frac{d}{dt}(m\dot{x}_1) = 0 \\ -kx_2 - 4k(x_2 - x_1) - \partial/\partial t(m\dot{x}_2) = 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} -5kx_1 + 4kx_2 - m\ddot{x}_1 = 0 \\ -5kx_2 + 4kx_1 - m\ddot{x}_2 = 0 \end{array}\right] \Rightarrow \ddot{\vec{x}} + \left(\sqrt{\frac{k}{m}}\right)^2 \left(\sqrt{A}\right)^2 \vec{x} = 0 \qquad A = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

?

< Functions of Matrices > ($\sqrt{A}$?)

* characteristic polynomial of A : $P_A(\lambda) = \det(\lambda I - A)$.

Suppose $\underbrace{\sqrt{\lambda}}_{\text{Our Goal}} = f(\lambda) = P_A(\lambda) \cdot q(\lambda) + r(\lambda)$

Cayley - Hamilton Theorem : $P_A(A) = \vec{0}$

$\Rightarrow \sqrt{A} = f(A) = r(A)$

$\boxed{\text{E.g.}}$ $A = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$

$$\sqrt{1} = f(1) = \cancel{P_A(1)}^{0}\, q(1) + (a \cdot \lambda + b)\Big|_{\lambda=1}$$
$$\sqrt{9} = f(9) = \cancel{P_A(9)}^{0}\, q(9) + (a\lambda + b)\Big|_{\lambda=9}$$

* $r(\lambda)$ is 1^{st} order $\because$ $P_A(\lambda)$ is 2^{nd} order

$\Rightarrow r(\lambda) = \frac{1}{4}\lambda + \frac{3}{4} \Rightarrow \sqrt{A} = \frac{1}{4}A + \frac{3}{4} \cdot I = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

< Homework #1 >

what is $\sin\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\cos\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$? $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

① sin

$P_A(\lambda) = \lambda^2 \Rightarrow \sin\lambda = f(\lambda) = P_A(\lambda)\, q(\lambda) + r(\lambda) = \lambda^2 q(\lambda) + C\lambda + D$

$\Rightarrow \cos\lambda = 2\lambda\, q(\lambda) + \lambda^2 q'(\lambda) + C$

$\Rightarrow D = 0,\ C = 1 \Rightarrow f(A) = \sin\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

② cos

$P_A(\lambda) = \lambda^2 \Rightarrow \cos\lambda = f(\lambda) = P_A(\lambda)\, q(\lambda) + r(\lambda) = \lambda^2 q(\lambda) + C\lambda + D$

$\Rightarrow -\sin\lambda = 2\lambda\, q(\lambda) + \lambda^2 q'(\lambda) + C$

$\Rightarrow C = 0,\ D = 1 \Rightarrow f(A) = \cos\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Solution form of given example. : $\vec{X}(t) = \cos \underbrace{\begin{pmatrix} 2w_0 t & -w_0 t \\ -w_0 t & 2w_0 t \end{pmatrix}}_{B} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

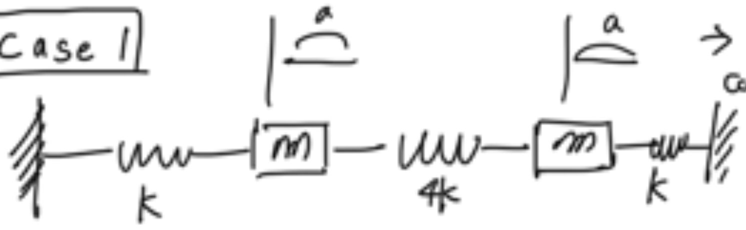
$$P_B(\lambda) = \det(\lambda I - B) = (\lambda - w_0 t)(\lambda - 3w_0 t)$$

$$\cos\lambda = f(\lambda) = P_B(\lambda)\, q(\lambda) + r(\lambda) = (\lambda - w_0 t)(\lambda - 3w_0 t)\, q(\lambda) + P\lambda + Q$$

$$\Rightarrow \begin{aligned} \cos w_0 t &= w_0 t \cdot P + Q \\ \cos 3w_0 t &= 3w_0 t \cdot P + Q \end{aligned} \quad \Rightarrow \quad \begin{aligned} P &= \frac{\cos 3w_0 t - \cos w_0 t}{2w_0 t} \\ Q &= \frac{1}{2}\left(3\cos w_0 t - \cos 3w_0 t\right) \end{aligned}$$

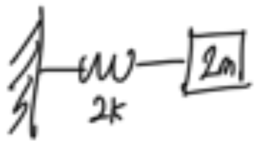
Wave Lecture 3 (20210907)

$\boxed{\text{Case 1}}$



→ initial conditions. $\vec{X}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$

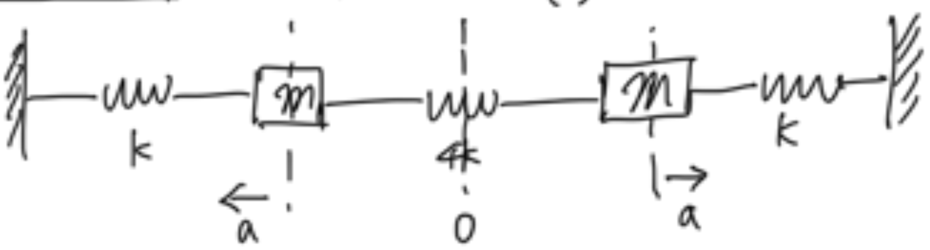
⇒ Intuition : No force by "4k" spring. (two masses synchronized)

⇒ Transparent "4k" ⇒ 

< In-phase movement >

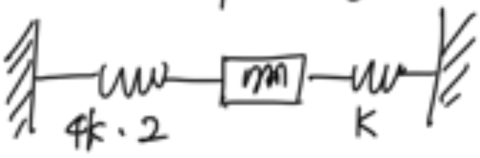
⇒ Frequency $= \omega_1 = \sqrt{k/m} = \omega_0$

$\boxed{\text{Case 2}}$: $x_1(0) = a, \; x_2(0) = -a$



< out of phase movement >

⇒ Intuition : Symmetric movement via point "O"

⇒ Identical to 

⇒ Frequency $= \omega_2 = \sqrt{(4k \cdot 2 + k)/m} = 3\sqrt{k/m} = 3\omega_0$

$\boxed{\text{HW}}$

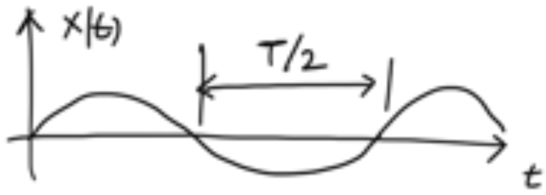
Q) Can we realize higher frequencies just by adjusting Initial Conditions?

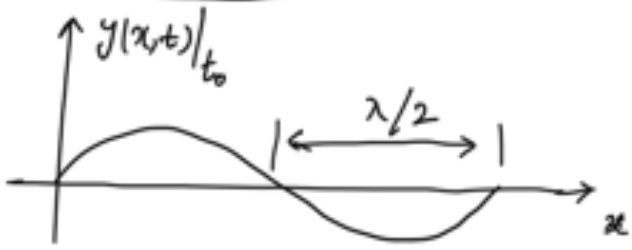
A) No. why ?

$\boxed{\infty \text{ DOF : lattice system.}}$

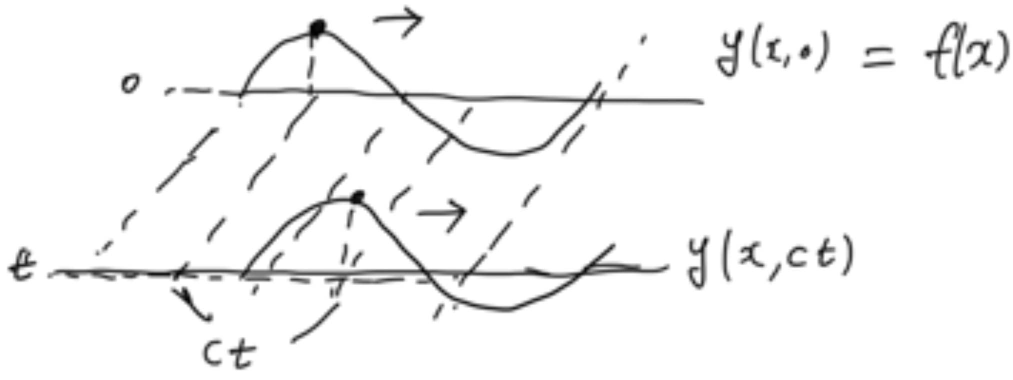
· Terminologies. :

<u>Temporal terms</u> $\longleftrightarrow$ <u>Spatial terms</u>





1) T (period) $\longleftrightarrow$ λ (wavelength)



$$\Rightarrow y(x, ct) = y(x-ct, 0) = \underline{\underline{f(x-ct)}}$$

$$\begin{cases} f(x-ct) : \vec{x} \text{ direction propagation} \\ f(x+ct) : -\vec{x} \quad " \quad " \end{cases}$$

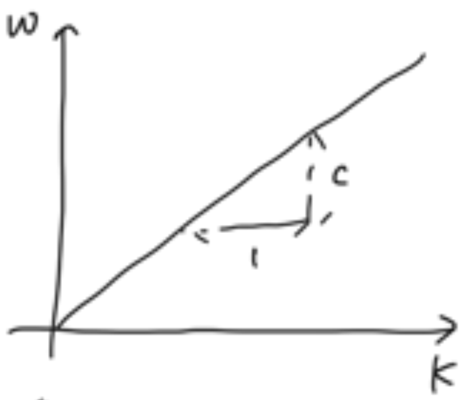
"Phase"

$$c = \frac{\lambda}{T} = \frac{\omega}{2\pi} \cdot \lambda = \omega / k$$

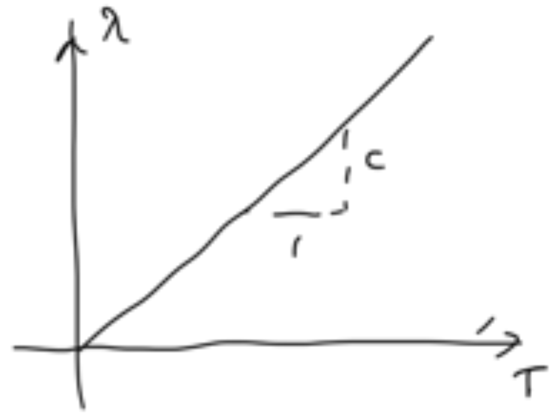
$\boxed{k}$ is # of wave in 2π (unit) length

$\underline{k = 2\pi/\lambda.}$

2) ω (angular frequency) $\longleftrightarrow$



$\approx$



"k-ω curve" : Dispersion Curve.

Wave Lecture 4 (20210909)

· 1D Wave Equation

① For $f(kx-\omega t)$: right going

$$\frac{\partial f}{\partial t} = \frac{\partial \phi}{\partial t}\frac{\partial f}{\partial \phi} = (-\omega) f'$$

$$\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial \phi} = k f'$$

$$\Rightarrow k f' c - \omega f' = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right] f = 0$$

② for $f(kx+\omega t)$: left going.

$$\partial f/\partial t = \omega f'$$

$$\partial f/\partial x = k f'$$

$$\Rightarrow k f' c - \omega f' = 0$$

$$\Rightarrow \left[\partial/\partial t - c\,\partial/\partial x\right] f = 0$$

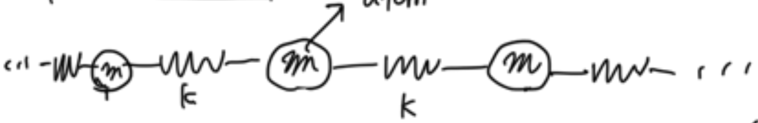
From ① and ②,

$$\left(\partial/\partial t + c\cdot \partial/\partial x\right)\left(\partial/\partial t - c\,\partial/\partial x\right) f = 0$$

$$\Rightarrow \left(\partial^2/\partial t^2 - c^2 \partial^2/\partial x^2\right) f = 0 \quad \Leftrightarrow \quad \underline{\left(\partial^2/\partial x^2 - \frac{1}{c^2}\partial^2/\partial t^2\right) f = 0}$$

1 Dimensional Wave Equation.

Lattice

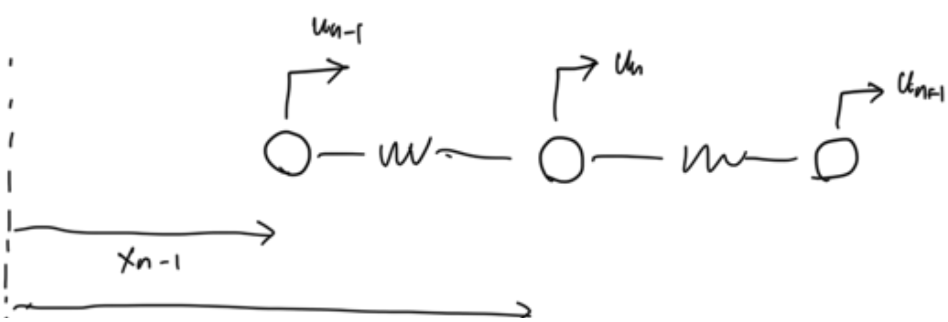


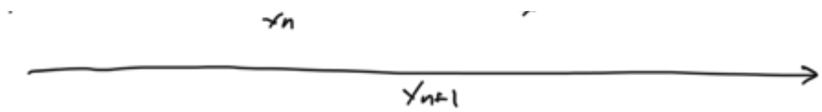
(lattice)

< Mon~~o~~atomic lattice>

types of mass

1. Mono
2. Di
3. Tri





$\boxed{E.O.M}$ $m\ddot{u}_n = +k(u_{n+1} - u_n) - k(u_n - u_{n-1}) = -k(2u_n - u_{n-1} - u_{n+1})$

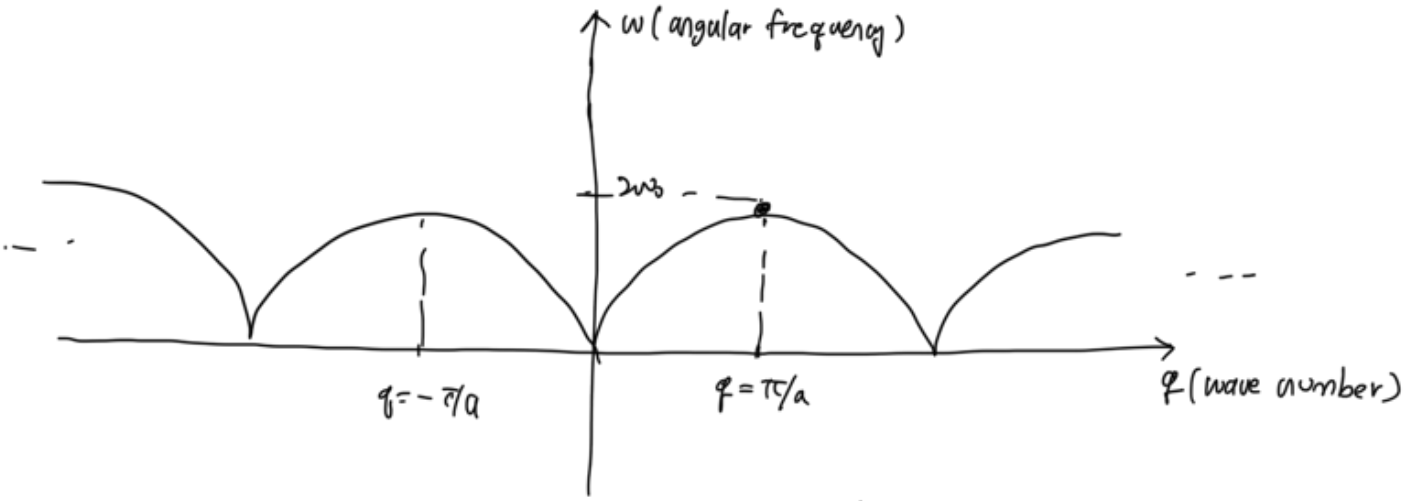
Let $u_n = A e^{j(q x_n - \omega t)}$ (q = wavenumber, A is same for all ∵ monatomic).

Since $x_n(t) = x_n + u_n(t)$, where x_n is equilibrium constant, $x_n = na$

$\Rightarrow -m\omega^2 e^{jqna} = +k(e^{jqna} \cdot 2 - e^{jq(n-1)a} - e^{jq(n+1)a})$

$\Rightarrow m\omega^2 = k(2 - 2\cos qa) = 4k(1 - \cos qa) = 4k\sin^2(qa/2)$

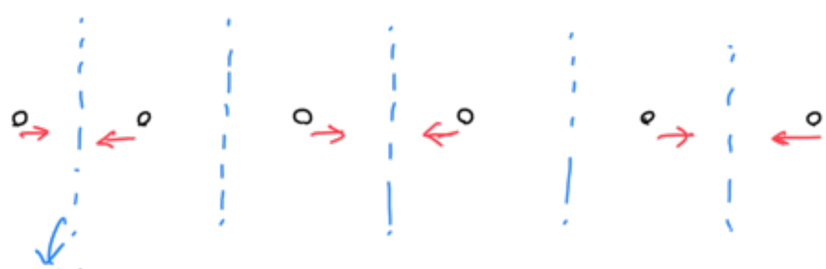
$\Rightarrow \omega = 2\sqrt{\frac{k}{m}}\left|\sin\left(\frac{qa}{2}\right)\right| = 2\omega_0\left|\sin\left(\frac{qa}{2}\right)\right|$ $\left(\omega_0 = \sqrt{\frac{k}{m}}\right)$



(1) when $q = \pi/a$, $u_n = A e^{j(qna - \omega t)} \to \varphi_n$
$u_{n+1} = A e^{j(q(n+1)a - \omega t)} \to \varphi_{n+1}$

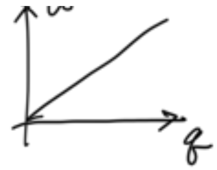
$\varphi_{n+1} - \varphi_n = qa = \pi.$

$\Rightarrow \begin{pmatrix} u_n = -u_{n+1} \\ u_n = -u_{n-1} \end{pmatrix}.$



nodal points. → no transport of wave energy.

(2) $\underset{\overset{\shortparallel}{0}}{\overset{\uparrow}{qa}} \simeq 0 \Rightarrow \omega = 2\omega_0 \left| \sin\left(\frac{qa}{2}\right) \right| \approx q\omega_0 a.$



phase speed

$= \text{slope}$

$= \omega_0 a$

$= \sqrt{\frac{K}{m}}\, a$

(3) $\max(\omega) = 2\omega_0 = \sqrt{4K/m}$ Q) why $4K$?

(4) $\omega(q) = \omega(-q)$ (symmetricity) : [right & left going waves has the same information]

Wave Lecture 5 (20210914)

Review of Compressible "Fluid Mechanics"

* Extensive & Intensive properties.

$B \longrightarrow \beta$

normalized by mass

$\Rightarrow$

	B	β
mass	m	1
momentum	$m\vec{V}$	$\vec{V}$
Energy	$\underbrace{E}_{\text{internal}} + \underbrace{\frac{1}{2}mV^2}_{\text{kinetic}}$	$e + \frac{1}{2}V^2$

* Conservation Laws.

· Reynolds Transport Theorem (RTT)

The rate of change of an extensive property (B)

= time rate of change of B within C.V. + net rate of flow B through C.S.

$$\Rightarrow \frac{dB}{dt} = \underbrace{\int_{\Omega}}_{\text{control volume}} \frac{\partial}{\partial t}(\rho\beta)\,dV + \underbrace{\int_{\partial\Omega}}_{\text{Control surface}} (\rho\beta)\,\vec{v}\cdot\vec{n}\,dA.$$

$$\Rightarrow \frac{d}{dt}\int_{\Omega}(\rho\beta)\,dV = \int_{\Omega}\frac{\partial}{\partial t}(\rho\beta)\,dV + \int_{\partial\Omega}(\rho\beta)\underbrace{\vec{V}_b}_{\partial\Omega \text{ velocity}}\cdot\vec{n}\,dA + \int_{\partial\Omega}(\rho\beta)\underbrace{\vec{V}_r}_{\text{fluid velocity w.r.t. } \partial\Omega}\cdot\vec{n}\,dA$$

① Continuity ($B = m$)

$$\frac{dB}{dt} = \int_\Omega \frac{\partial}{\partial t}(\rho \cdot 1)\, dV + \int_{\partial\Omega} (\rho \cdot 1)\, \vec{u}\cdot\vec{n}\, dA = \int_\Omega \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{u}) \right) dV$$

$$\| $$

$\frac{dm}{dt} = 0$ (∵ mass of system is conserved)

(∵ Gauss divergence theorem $\int_\Omega \nabla\cdot\vec{F}\, dV = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, dA$)

$\Rightarrow \partial\rho/\partial t + \nabla\cdot(\rho\vec{u}) = 0$ (∵ if $\int_p f\, dV = 0 \Rightarrow f = 0$ by localization theorem)

② Momentum. $B = m\vec{u}$, $\beta = \vec{u}$ $\vec{B} = (B_1, B_2, B_3)$

$$\frac{dB_i}{dt} = \int_\Omega \frac{\partial}{\partial t}(\rho \cdot u_i)\, dV + \int_{\partial\Omega} (\rho \cdot u_i)\, \vec{u}\cdot\vec{n}\, dA \quad \text{for } i = 1, 2, 3$$

$$\|$$

$\frac{d\vec{P}}{dt} = \vec{F}$

(net force acting on $\partial\Omega$) = normal force + shear force

$$\|$$

pressure is inwards

$$-\int_{\partial\Omega} P\, \vec{e_i}\cdot\vec{n}\, dA$$

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$$\Rightarrow \int_\Omega \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{\partial\Omega} [\rho u_i\, \vec{u}\cdot\vec{n} + P\vec{e_i}\cdot\vec{n}]\, dA = 0$$

$$\Rightarrow \int_\Omega \left[\frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(\rho u_i \vec{u}) + \nabla\cdot(P\vec{e_i}) \right] dV = 0$$

$$\Rightarrow \int_\Omega \frac{\partial}{\partial t}(\rho u_i) + u_i \nabla\cdot(\rho\vec{u}) + \rho\vec{u}\cdot\nabla u_i + \frac{\partial P}{\partial x_i}\, dV = 0$$

$$\left(∵ \nabla\cdot(P\vec{e_1}) = \nabla\cdot(P, 0, 0) = \frac{\partial P}{\partial x_1} \right)$$

$$\cancel{\frac{\partial \rho}{\partial t}} u_i + \rho\frac{\partial u_i}{\partial t} + \cancel{u_i \nabla\cdot(\rho\vec{u})} + \rho\vec{u}\,\nabla u_i = -\frac{\partial P}{\partial x_i}$$

∵ Continuity equation.

$$\Rightarrow -\partial P/\partial x_i = \rho\frac{\partial u_i}{\partial t} + \rho\vec{u}\,\nabla u_i = \rho\left(\frac{\partial u_i}{\partial t} + \vec{u}\,\nabla u_i \right)$$

③ State equation. (thermodynamic process)

$$\underline{p = p(\rho, T)} \quad \text{or} \quad p = p(\rho, s)$$

$$\underline{p - p_0} = \left.\frac{\partial p}{\partial \rho}\right|_{s=s_i} (\rho - \rho_0) + \cancel{\text{Taylor}} \to 0 + \left.\frac{\partial p}{\partial s}\right|_{\rho=\rho_i} (s - s_0) + \cancel{\text{Taylor}} \to 0$$

$$\parallel$$

$$p' = \left.\frac{\partial p}{\partial \rho}\right|_{s_0} \rho' + \underbrace{\left.\frac{\partial p}{\partial s}\right|_{\rho_0} s'}_{\left(\text{=0 for isentropic process}\right)} \Rightarrow \underline{p' = \left.\frac{\partial p}{\partial \rho}\right|_{s_0} \rho'} \quad ①$$

For adiabatic process,

$$p/\rho^{\gamma} = p_0/\rho_0^{\gamma} \Rightarrow \partial p/\partial \rho = \frac{p_0}{\rho_0^{\gamma}} \gamma \rho^{\gamma-1} = p\gamma \frac{\rho^{\gamma}}{\rho_0^{\gamma}\cdot\rho} = \frac{\gamma p}{\rho} = \frac{\gamma \rho R T}{\rho}$$

From ① $\Rightarrow$ $\underline{p' = \gamma R T \rho'}$ //

Note : Isentropic process assumes ideal state from adiabatic process.

Note : $\sqrt{\gamma R T}$ = speed of sound (c)

Wave Lecture 6 (20210916)

- Summary

Linearization

$$\begin{cases} \rho = \rho_0 + \rho' \\ u = u_0 + u' \end{cases} \Rightarrow$$

⟨Linearized⟩. • 숙제 • 다음주 휴강.

① $\partial\rho/\partial t + \nabla\cdot(\rho u) = 0$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot u' = 0 \quad ①$$

② $\rho\left(\partial u/\partial t + u\cdot\Delta u\right) = -\nabla p$

$$\Rightarrow \quad (\rho_0 + \rho')\left(\frac{\partial u'}{\partial t} + \cancel{u'\cdot\nabla u'}^{\,0}\right) \approx \rho_0 \frac{\partial u'}{\partial t} = -\nabla p' \quad ②$$

$\rho' \to 0$

③ $p' = \left(\sqrt{\gamma R T}\right)^2 \rho'$ (isentropic process)

⟨speed of sound : Laplace⟩ (Isentropic)

$$\partial ①/\partial t \Rightarrow \frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\nabla\cdot u' = 0$$

$$\Rightarrow \nabla^2 p' - \frac{\partial}{\partial t^2}\left(\frac{1}{\sqrt{\gamma R T}}\right)^2 p' = 0$$

$\nabla\cdot\nabla = \nabla^2$ Laplacian.

$$\nabla\cdot ② \Rightarrow \rho_0 \nabla\cdot\frac{\partial}{\partial t}u' = -\nabla\cdot(\nabla p')$$

$$\Rightarrow \nabla^2 p' - \left(\frac{1}{\sqrt{\gamma R T}}\right)^2 \frac{\partial^2}{\partial t^2} p' = 0$$

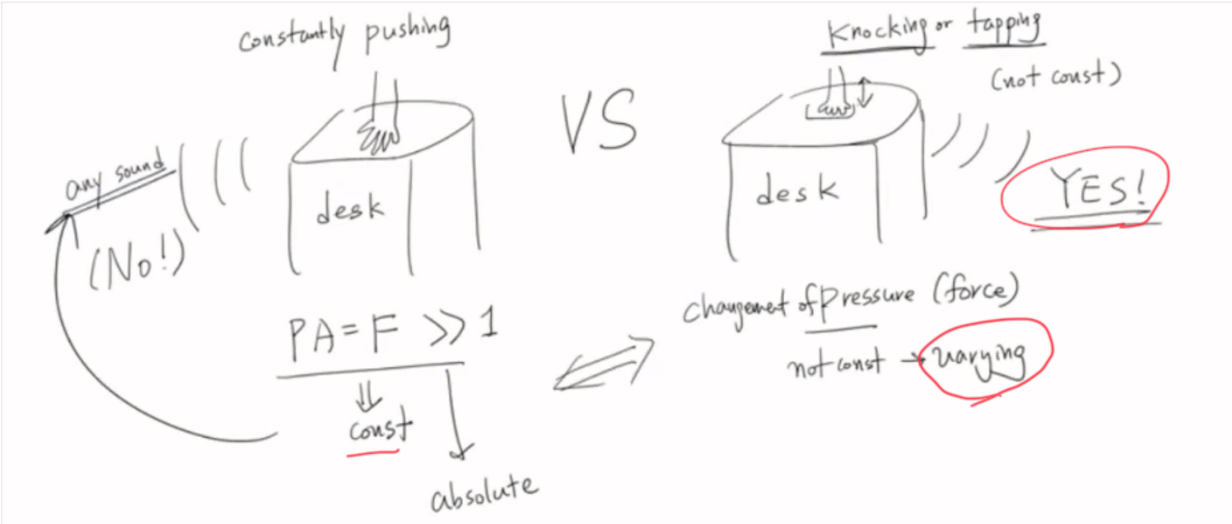
Waves in compressible fluid = "Sound" (Acoustics)

⟨Speed of sound : Newton⟩ (Isothermal)

$$p = p(\rho, T) \Rightarrow p' = \left.\frac{\partial p}{\partial \rho}\right|_{T_0} \rho' + \cancel{\left.\frac{\partial p}{\partial T}\right|_{\rho_0} T'}^{\,0} \rightarrow p' = \left.\frac{\partial p}{\partial \rho}\right|_{T_0} \rho'$$

Since $p = \rho R T$, $\partial p/\partial \rho = RT \Rightarrow p' = \left(\sqrt{RT_0}\right)^2 \rho'$ $\quad \sqrt{RT_0} \approx 290 \rightarrow$ Wrong !

Q1. What is sound ?



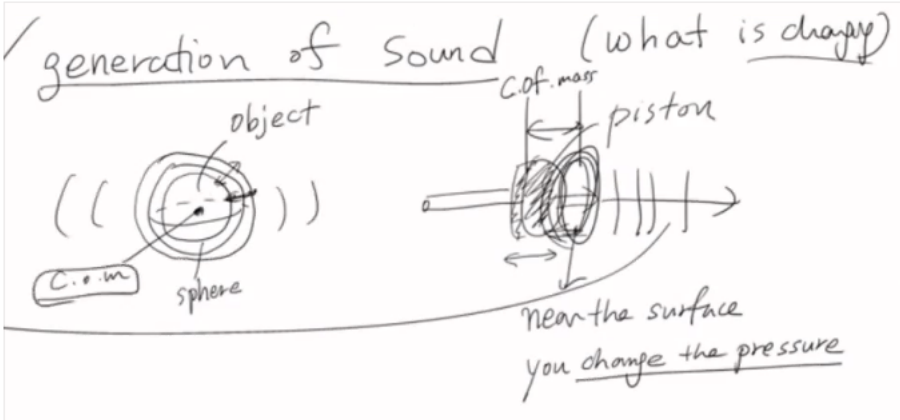
$$p = \underbrace{p_0}_{\text{mean}} + \underbrace{p'}_{\text{perturbed}}$$ → Sound = Acoustic wave

Q2. What Differential Equation governs the propagation of sound?

Lin Cont.
Lin Euler
State Eq. (isentropic)

→ $\nabla^2 p' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} p' = 0$ —— Eq ①

Q3. How does the sound propagate? : Change.



Alternative way of expression.

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct \qquad (\text{for 1 dimension : X axis}).$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \Rightarrow \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) \Rightarrow \frac{\partial^2}{\partial t^2} = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \quad \text{and} \quad \frac{\partial}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial \eta^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \xi^2}\right)$$

$\Rightarrow$ From equation ①, assuming that $p'(x,t) = \tilde{p}(\xi, \eta)$,

for 1 dimensional case. $\frac{\partial^2}{\partial x^2} p' = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} p'$

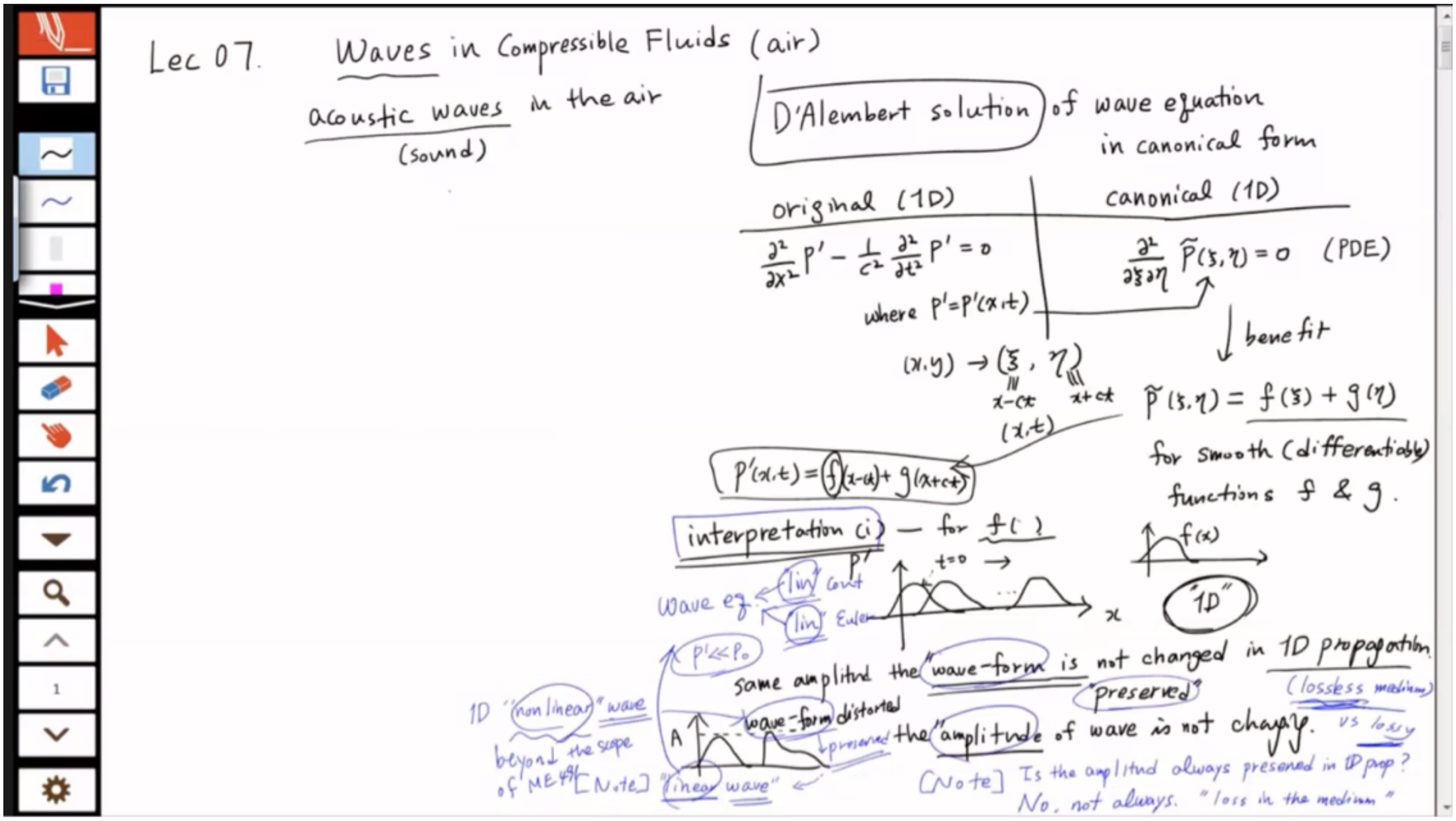
$$\Rightarrow \left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)\tilde{p}(\xi,\eta) = \frac{1}{c^2}\cdot c^2\left(\frac{\partial^2}{\partial \eta^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \xi^2}\right)\tilde{p}(\xi,\eta)$$

$$\Rightarrow 4\frac{\partial^2}{\partial \xi \partial \eta}\tilde{p}(\xi,\eta) = 0 \Rightarrow \frac{\partial^2}{\partial \xi \partial \eta}\tilde{p}(\xi,\eta) = 0$$

→ Canonical form of wave equation
→ Easy to solve PDE.

general solution form is. $\tilde{p}(\xi,\eta) = f(\xi) + g(\eta)$ (D'Alembert Solution).

Wave Lecture 7 (20210928)



interpretation (2) D'Alembert sol'n

preparation ① "acoustic potential" ϕ ⟶ "preparation ②" "wave eq" in terms of ϕ — not p'

Recall the velocity potential Φ.

For an irrotational velocity field, $\nabla \times u \equiv 0$ (irrotational)

There exists a scalar fn Φ such that $u_{irrot} = \nabla \Phi$ (scalar fn.)

$u = u_{irrot} + u_{soled}$

By using a vector identity, $\underbrace{\nabla \times \nabla h}_{\text{vector}} \equiv \underbrace{0}_{\text{0-vector}}$ (h: scalar fn)

$0 \equiv \nabla \times u_{irrot} = \nabla \times \nabla \Phi \equiv 0$

single scalar fn $\xrightarrow{\text{express}}$ a vector field.

perturbed quantity

$u' = \nabla \phi$ (scalar)

"what is changing"

perturbed velocity potential → acoustic potential

(i) Linearized Euler eq.

$\rho_0 \frac{\partial u'}{\partial t} = -\nabla p' \quad (u_0 \equiv 0)$

$u' = \nabla \phi$

$\rho_0 \frac{\partial}{\partial t}(\nabla \phi) = -\nabla p'$

$\nabla\left(\rho_0 \frac{\partial}{\partial t}\phi\right) = \nabla(-p')$

$\Rightarrow \boxed{p' = -\rho_0 \frac{\partial \phi}{\partial t}}$ relation btw p' & ϕ

$\boxed{u' = \nabla \phi}$ relation btw u' & ϕ

solution ϕ → ϕ → p', u' (acoustics)

∇^2

(ii) Linearized continuity eq.

$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u' = 0 \xrightarrow{\text{state eq.}} \frac{1}{c^2}\frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot (\nabla \phi) = 0$

$\rightarrow \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\rho_0 \frac{\partial \phi}{\partial t}\right) + \rho_0 \nabla^2 \phi = 0$

Wave eq.

$\nabla^2 p' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p' = 0$

$\rightarrow \boxed{\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0}$ wave eq. in terms of ϕ

interpretation ② of D'Alembert solution (1D)

$u' = \nabla \phi$

$p' = -\rho_0 \frac{\partial \phi}{\partial t}$

$\xRightarrow{\text{"1D"}}$

$u' = \frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial x}(\underbrace{x-ct}_{\xi})$

$p' = -\rho_0 \frac{\partial \phi}{\partial t}$

$\phi = \underbrace{f(x-ct)}_{\text{"right-going" wave}} + \underbrace{g(x+ct)}_{\text{left-going}}$

$\frac{\partial \xi}{\partial x} \frac{\partial f(\xi)}{\partial \xi} = 1 \cdot f'(\xi)$

$-\rho_0 \frac{\partial f}{\partial t}(x-ct) \qquad \frac{\partial x}{\partial t} = -c$

$= (-)\rho_0 \cdot (-c) \cdot f'$

combining

$u' = f'(\xi)$

$p' = \rho_0 c \, f'(\xi)$

$\Rightarrow p' = \rho_0 c \, u'$

phy. var. — "material" properties — phy. var.

$\rho_0 c = Z$ " "

$\frac{p'}{u'}$ = acoustic impedance

$\times \frac{1}{\text{area}}$

recall $\frac{F}{u}$ = "mechanical impedance"

how resistive

HW : Find relation btw p', u' for left going wave.

$u' = \partial\phi/\partial x = \frac{\partial g}{\partial x}(x+ct)$

$p' = -\rho_0 \partial\phi/\partial t = -\rho_0 c \, g' \qquad \frac{\partial \eta}{\partial t} = c$

$Z = -\rho_0 c.$

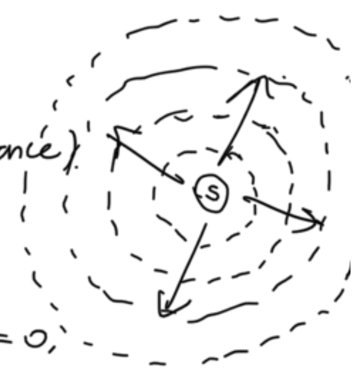
<Concentrate on sound propagation.>

① 1D propagation.

$$p'(x,t) = f(x-ct) + g(x+ct) \quad \text{and} \quad \phi(x,t) = f_1(x-ct) + g_1(x+ct)$$

② 3D propagation.

$p'(r,\cancel{\theta},\cancel{\phi},t) = p'(r,t)$: function of r, t (radial distance)



$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)p' = 0 \xrightarrow{\text{cartesian}} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)p' = 0,$$

$$\xrightarrow{\text{spherical}} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \cancelto{0}{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)} + \cancelto{0}{\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}.$$

<Homework> : derivation.

W.E in 3 dim : $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)p'(r,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p'(r,t) = 0$.

$$\Rightarrow \frac{1}{r^2}\left(2r\frac{\partial}{\partial r}\right)p' + \frac{\partial^2}{\partial r^2}p' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p' = 0$$

$$\Rightarrow \frac{\partial^2}{\partial r^2}p' + \frac{2}{r}\frac{\partial}{\partial r}p' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p' = 0$$

$$\Rightarrow \frac{1}{r}\left(r\frac{\partial^2}{\partial r^2} + 2\frac{\partial}{\partial r}\right)p' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}(rp')\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p' = 0 \Rightarrow \underbrace{\frac{\partial^2}{\partial r^2}(rp') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rp') = 0}_{\text{1 dimensional equation respect to } rp'}.$$

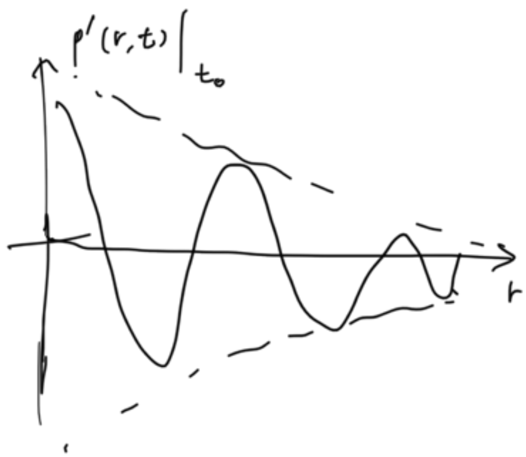
D'Alembert

$$\Rightarrow rp' = f(r-ct) + g(r+ct)$$

$$\Rightarrow \underbrace{p' = \frac{f(r-ct)}{r}}_{\text{far away from origin}} + \underbrace{\frac{g(r+ct)}{r}}_{\text{closer to origin}}$$

far away from origin ⇒ outgoing wave

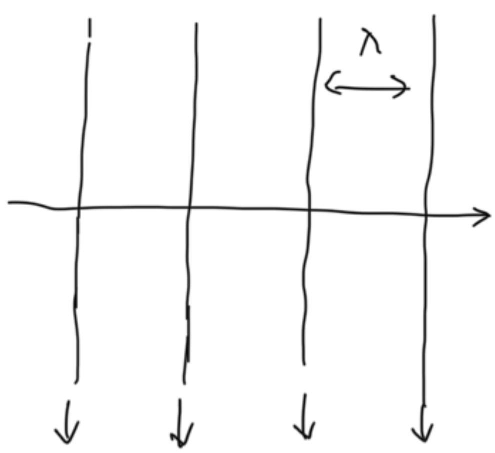
closer to origin ⇒ in-coming wave



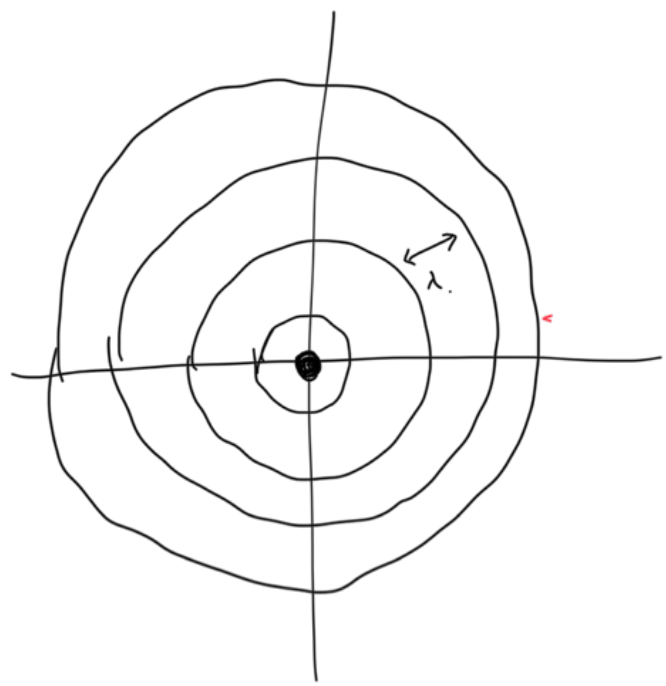
1. wave-form is preserved but scaled ($1/r$)
2. Amplitude is changing depending on "r"

$\boxed{\text{Problem}}$

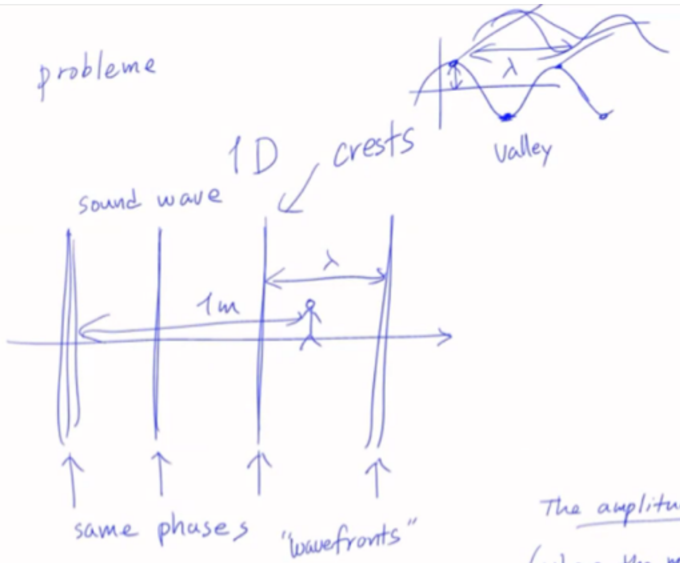
1D sound wave.

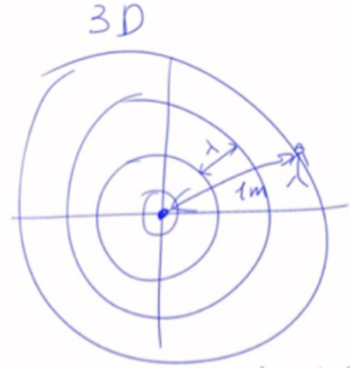


same phases : wave fronts.



probleme





The amplitude of pressure perturbation at a distance 1m from a sound source (where the medium is air) was $1\ N/m^2$ and the frequency of the wave was 500 Hz.

Calculate the amplitude of particle velocity at 0.1m from the source

$$\underbrace{\phi = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}}$$

$$\underline{p' = \frac{h(r-ct)}{r}}$$ form.

Linearized equations

$$\begin{cases} p' = -\rho_0 \dfrac{\partial \phi}{\partial t} \\ u' = \nabla \phi \end{cases}$$

$$\partial \rho' / \partial t + \rho_0 \nabla \cdot u = 0 \quad \text{(continuity)}$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot (\nabla \phi) = 0$$

$$\Rightarrow \underline{\nabla^2 \phi - \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0} \,/\!/$$

$$u' = \nabla \left(\frac{f(r-ct)}{r} \right) \quad \text{and} \quad p' = -\rho_0 \frac{\partial}{\partial t} \left(\frac{f(r-ct)}{r} \right).$$ (assume right-going wave)

$$\Rightarrow u' = \frac{\partial}{\partial r} \left(\frac{f(r-ct)}{r} \right) \hat{r}$$ (no components in θ, φ).

$$= \frac{f'(r-ct) \cdot r - f(r-ct)}{r^2} \hat{r} = \frac{1}{r} \frac{\partial}{\partial r} f(r-ct) - \frac{1}{r^2} f(r-ct).$$

$$p' = -\rho_0 \frac{1}{r} (-c) f'(r-ct). = \frac{\rho_0 c}{r} \frac{\partial}{\partial t} f(r-ct)$$

① Since $f(x) = e^{jkx}$ so that $f(r-ct) = \underbrace{e^{j(k(r)-wt)}}$ $\quad kc = w$

$\Rightarrow u' = \frac{1}{r} jk\, f(r-ct) - \frac{1}{r^2} f(r-ct)$ and $p' = \frac{\rho_0 c}{r} jk\, f(r-ct)$

$$\Rightarrow \underset{\substack{\downarrow \\ \text{Acoustic} \\ \text{impedance.}}}{z} = \frac{p'}{u'} = \frac{\frac{1}{r}\cdot k \cdot \rho_0 c\, j}{-\frac{1}{r^2} + \frac{1}{r}\cdot k \cdot j} \; \frac{\cancel{f(r-ct)}}{\cancel{f(r-ct)}} = \left\{ \frac{rk\, j}{-1 + rk\cdot j} \right\} \rho_0 c$$

$$\Rightarrow |z| = \frac{|p'|}{|u'|} = \frac{rk}{\sqrt{1+(rk)^2}}\rho_0 c = \frac{rw}{\sqrt{c^2 + (rw)^2}}\rho_0 c$$

since $|p'| = 1\ N/m^2$ and $f = 500\ Hz$

since wave form does not change, $|p'|$ at $0.1\ m = 10 N/m^2$ with same angular frequency $f = 500\ Hz \to w = 2\pi \cdot 500 \cdot$ rad/s.

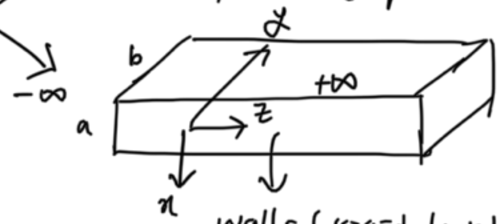
$$\Rightarrow |z| = \frac{|10|}{|u'|_{r=0.1m}} = \frac{0.1\cdot 2\pi\cdot 500}{\sqrt{(343\ m/s)^2 + (0.1\cdot 2\pi\cdot 500)^2}} \underset{kg/m^3}{(1.225)} \underset{m/s}{(343)} = 283.796$$

$$\Rightarrow |u'|_{r=0.1m} = 0.035\ (m/s) = \underline{35.24\ (mm/s)}.$$

Wave Lecture 8 (20210930)

3D sound propagation ↗ free-space (spherical symmetric) $\Rightarrow p'(r,t)$

↘ E.g. Rectangular, uniform cross section,



walls (rigid boundary): non-penetration condition

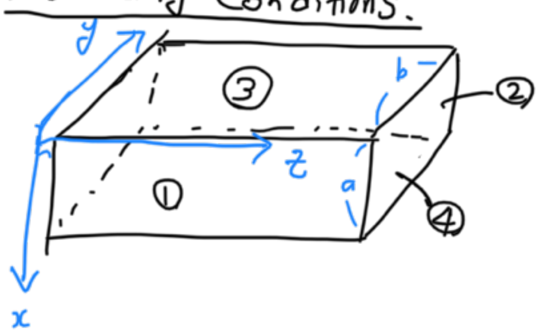
$u_n = 0$

$\downarrow = 0$

① G.E. (Governing Equation)

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = 0 \qquad (\vec{u}' = \nabla \phi)$$

② Boundary Conditions.



① $u'_y \big|_{y=0} = 0$ ② $u'_y \big|_{y=b} = 0$ $0 \le x \le a$, $-\infty < z < \infty$

③ $u'_x \big|_{x=0} = 0$ ④ $u'_x \big|_{x=a} = 0$ $0 \le y \le b$, $-\infty < z < \infty$

Note: $u'_x = \frac{\partial \phi}{\partial x}$, $u'_y = \frac{\partial \phi}{\partial y}$ $\Rightarrow$

① $\frac{\partial \phi}{\partial y}\Big|_{y=0,b} = 0 \quad (0 \le x \le a)$

② $\frac{\partial \phi}{\partial x}\Big|_{x=0,a} = 0 \quad (0 \le y \le b)$

4 B.C.s.

$\boxed{\text{Solve}}$

$$\phi(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,e^{-j\omega t}$$

$$\Big\{ X''YZ + XY''Z + XYZ'' - \underbrace{\frac{1}{c^2}(-j\omega)^2}_{\frac{\omega^2}{c^2} = k^2} XYZ \Big\}\, \underbrace{e^{-j\omega t}}_{\neq 0} = 0$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0$$

(1) $\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} - k^2 = \underbrace{-\mu^2}_{\text{negative!}}$ ($\because$ to get sinusoidal form, we set negative)

x-only y, z only const.

$\Rightarrow \underbrace{X'' + \mu^2 X = 0}_{\text{ODE}} \Rightarrow X(x) = A\cos(\mu x) + B\sin(\mu x)$

+ B.C. w.r.t x

$\frac{\partial \phi}{\partial x}\Big|_{x=0,a} = 0$ for any $a \le y \le b$

$X'' = 0 \rightarrow$ linear

$\boxed{B=0}$ or $\cancel{\mu=0}$

$\mu a = n\pi \quad (n = 0, 1, 2, \cdots)$

$\downarrow$

$\mu_n = \frac{n\pi}{a}$

$\begin{cases} B\mu \cdot 1 = 0 \\ -A\mu \sin \mu a = 0 \\ \quad \neq 0 \ \neq 0 \end{cases} \Leftarrow \frac{\partial X}{\partial x}\Big|_{x=0,a} = 0 \Rightarrow \begin{cases} -A\mu \sin\mu x + B\mu\cos\mu x \big|_{x=0} = 0 \\ -A\mu\sin\mu x + B\mu \cos\mu x \big|_{x=a} = 0 \end{cases}$

$$\therefore X(x) = \sum_m A_m \cos\left(\frac{m\pi}{a}x\right) \qquad \because \left.\partial\phi/\partial x\right|_{x=0,a} = 0$$

$$(2) \quad \frac{Y''}{Y} = -\frac{Z''}{Z} + \mu_m^2 - k^2 = -\nu^2 \Rightarrow Y(y) = \sum_n B_n \cos\left(\frac{n\pi}{b}y\right)$$

$$(3) \quad \frac{Z''}{Z} = \mu_m^2 + \nu_n^2 - k^2 = -\alpha_{mn}^2 \Rightarrow Z(z) = \sum_m \sum_n \left\{ C_{mn} e^{j\alpha_{mn} z} + D_{mn} e^{-j\alpha_{mn} z} \right\}.$$

From (1), (2), (3),

$$\phi(x,y,z,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\left\{ C_{mn} e^{j\alpha_{mn} z} + D_{mn} e^{-j\alpha_{mn} z} \right\}.$$

* Physical Interpretation.

$$\alpha_{mn} = \pm\sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}$$

$\nearrow \sqrt{///} > 0 \Rightarrow e^{j(\text{real})z} \Rrightarrow$

Propagating

$\searrow \sqrt{///} < 0 \Rightarrow e^{j(j(\text{real}))z} = e^{-(\text{real})z}$

evanescent

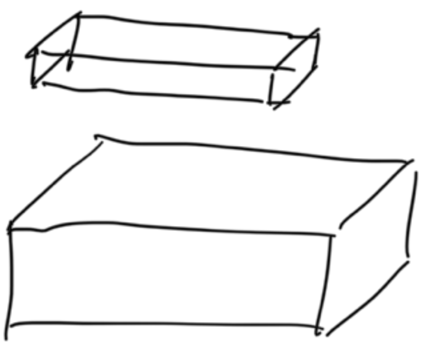
$$\left.\begin{array}{ll} \left(\frac{\omega}{c}\right)^2 - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) > 0 & : \text{Propagating} \\ < 0 & : \text{Evanescent.} \end{array}\right|$$

① $(m,n) = (0,0)$: plane wave mode

crests (wavefronts)

② Plane wave → always propagation inside waveguide.
For higher modes, it can become evanescent waves.

③ (a,b) → size of duct.



) → it matters!

Note: These are all in compressible fluids.

Wave Lecture 9 (20211005)

· Review of Incompressible Fluid Mechanics → Water waves.

① Cont : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \xrightarrow{\rho = \rho_0 \text{ (Constant)}} \underline{\nabla \cdot \vec{u} = 0}$

if irrotational flow , $u = \nabla \phi \Rightarrow \underline{\nabla^2 \phi = 0}$

② Euler Eq. : $\rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p - \rho g \ \vec{e_z}$

$\Rightarrow \partial u / \partial t + (u \cdot \nabla) u = -\nabla \left(\frac{p}{\rho} \right) - g \ \vec{e_z}$

since $\nabla (\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$

$(\vec{A} = \vec{B} = \vec{u} = \nabla \phi)$

$\Rightarrow \nabla (\nabla \phi \cdot \nabla \phi) = 2 \cdot (\nabla \phi \cdot \nabla) \nabla \phi + 2 \cdot \nabla \phi \times \cancelto{0}{(\nabla \times \nabla \phi)}$

$\Rightarrow \nabla (\vec{u} \cdot \vec{u}) = 2 (u \cdot \nabla) u$

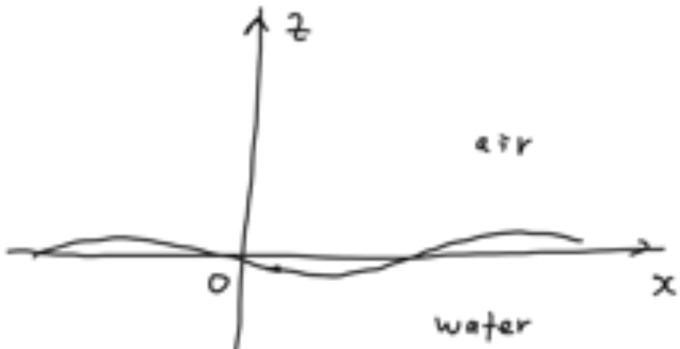
$\Rightarrow \frac{\partial u}{\partial t} + \frac{1}{2} \nabla |\vec{u}|^2 + \nabla \left(\frac{p}{\rho} \right) + g \vec{e_z} = 0$

$\Rightarrow \nabla \left(\frac{\partial}{\partial t} \phi + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} \right) = -\nabla (g z)$

$\Rightarrow \nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + g z \right) = 0$

HW #2. $\nabla \times \nabla \phi = 0$ prove it!

$$\nabla \times \nabla \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = 0 \hat{i} + 0 \hat{j} + 0 \hat{k} = \underline{\vec{0}}.$$



$$\Rightarrow \underline{\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + g z = \text{Const.}}$$

At $p = P_{ambient}$. undisturbed, neglect slope $\Rightarrow \frac{P_{amb}}{\rho} =$ Constant , $|\nabla \phi|^2 \sim 0$

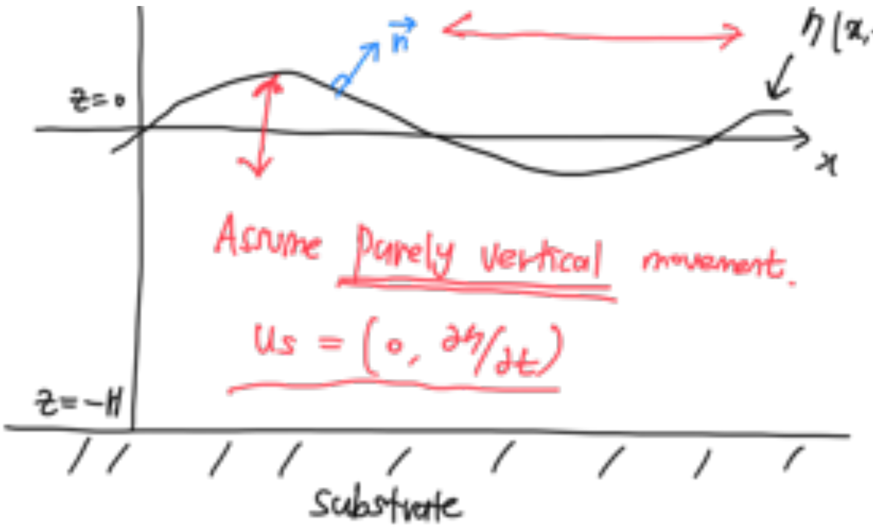
$$\Rightarrow \frac{\partial \phi}{\partial t} + \cancelto{0}{\frac{1}{2} |\nabla \phi|^2} + \frac{P_{absolute}}{\rho} + g z = \frac{P_{ambient}}{\rho}.$$

$$\Rightarrow \underline{\frac{\partial \phi}{\partial t} + \frac{P_{gauge}}{\rho} + g z = 0}$$

· $\boxed{\text{Surface gravity wave.}}$

z

Long wavelength → $\underline{\frac{\partial h}{\partial x} \ll 1}$



Assume purely vertical movement.

$U_s = (0, \partial\eta/\partial t)$

$\eta(x,t)$: Surface vertical deflection.

G.E : $\dfrac{\partial\phi}{\partial t} + \dfrac{P}{\rho} + gz = 0$

B.C : ① $u_z(z=-H)=0 \Rightarrow \boxed{\left.\dfrac{\partial\phi}{\partial z}\right|_{z=-H} = 0}$

② $\underbrace{\left.\vec{n}\cdot\vec{u}_p\right|_{z=\eta}}_{\text{fluid particle velocity normal to the surface}} = \underbrace{\vec{n}\cdot\vec{U}_s}_{\text{velocity of surface normal to it}}$

② ⇒ Surface $f(x,z,t)=0 \Rightarrow z-\eta(x,t)=0$

$$\left(\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\left(-\frac{\partial\eta}{\partial x}, 1\right)}{\|\nabla f\|}\right) \Rightarrow \left.\vec{n}\cdot\vec{u}\right|_{z=\eta} = \vec{n}\cdot\vec{u}_s$$

$$\Rightarrow \left(-\frac{\partial\eta}{\partial x}, 1\right)(u,w) = \left(-\frac{\partial\eta}{\partial x}, 1\right)\left(0, \frac{\partial\eta}{\partial t}\right)$$

Q) what if we don't neglect x direction of surface?

$$\Rightarrow -\frac{\partial\eta}{\partial x}u + w = \frac{\partial\eta}{\partial t} \qquad \left(w = \left.\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} + \underbrace{\eta\frac{\partial^2\phi}{\partial z^2} + \cdots \text{Higher order}\ldots}_{\text{Neglect} \to 0}\right)$$

Q) what if we include all Higher Order Terms?

$$\Rightarrow \left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t} + \underbrace{\frac{\partial\eta}{\partial x}}_{\ll 1}\cdot\frac{\partial\phi}{\partial x} \Rightarrow \boxed{\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t}}$$

Q) what if λ is small?

③ at $z=\eta$, guage pressure at interface $=0$ (surface tension X)

$$\frac{\partial\phi}{\partial t} + \cancel{\frac{P}{\rho}}^{\,0} + gz = 0 \Rightarrow \left.\frac{\partial\phi}{\partial t}\right|_{z=\eta} \simeq \left.\frac{\partial\phi}{\partial t}\right|_{z=0} = -g\eta$$

$$\Rightarrow \boxed{\left.\frac{\partial\phi}{\partial t}\right|_{z=0} = -g\eta}$$

Problem : $\phi(x,z,t) = \underbrace{f(z)}_{\text{depth term}}\sin(kx - w(k)\cdot t)$

initial Condition : $\eta(x, t=0) = a\cos kx$

① From continuity, $\nabla^2\phi = 0 \Rightarrow \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0 \Rightarrow \dfrac{d^2 f}{dz^2} - k^2 f = 0$

$\Rightarrow f(z) = Ae^{kz} + Be^{-kz}$.

$$\Rightarrow \phi(x,z,t) = (Ae^{kz} + Be^{-kz}) \sin(kx - w(k)\cdot t)$$

① $$\frac{\partial\phi}{\partial z}\Big|_{z=-H} = 0 \Rightarrow Ake^{-kH} - Bke^{+kH} = 0 \Rightarrow A = Be^{2kH} \to ①$$

② $$\frac{\partial\phi}{\partial z}\Big|_{z=0} = \frac{\partial\eta}{\partial t} \Rightarrow (Ake^{kz} - Bke^{-kz})\Big|_{z=0} \cancel{\sin(kx-w(k)t)} = \frac{A+B}{g} w^2(k)\, \cancel{\sin(kx-w(k)t)}$$

(by ③)

$$\Rightarrow (A-B)k = \frac{A+B}{g}\{w(k)\}^2$$

$$\Rightarrow w(k) = \sqrt{gk\,\frac{A-B}{A+B}} = gk\,\frac{e^{2kH}-1}{e^{2kH}+1} = \sqrt{gk\,\frac{e^{kH}-e^{-kH}}{e^{kH}+e^{-kH}}}$$

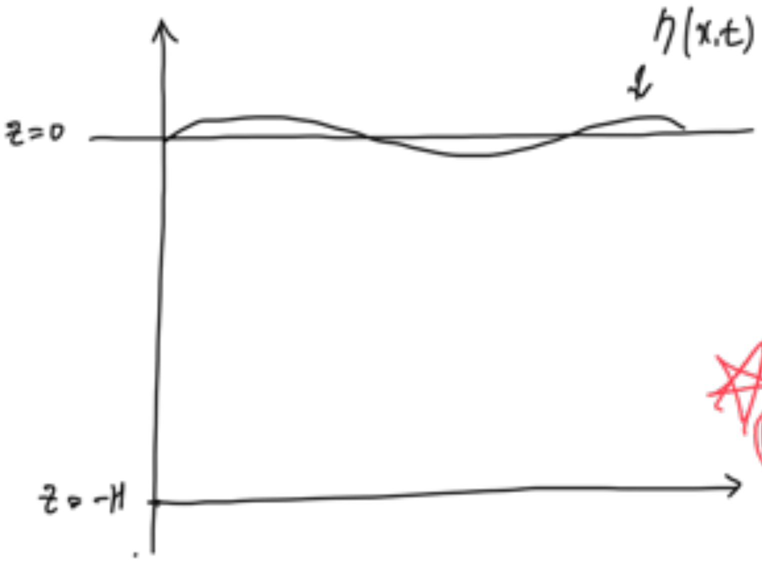
③ $$\frac{\partial\phi}{\partial t}\Big|_{z=0} = -g\eta \Rightarrow (Ae^{kz}+Be^{-kz})\{-w(k)\}\cos(kx-w(k)t)\Big|_{z=0} = -g\eta$$

$$\Rightarrow g\eta = (A+B)\, w(k)\cos(kx - w(k)t)$$

$$\Rightarrow \eta = \frac{A+B}{g}\, w(k)\cos(kx-w(k)t) \to ③$$

From ②, $w(k) = \sqrt{gk\tanh(kH)}$

Wave Lecture 10 (20211007)



at $t=0$ $\eta(x,t=0) = a\cos kx$. (I.C.)

Incompressible $\rightarrow \nabla\cdot\vec{u}=0$

Irrotational $\rightarrow$ there exists $\nabla\phi = \vec{u}$ $\Rightarrow$ $\underline{\nabla^2\phi=0}$

$\phi(x,z,t) = f(z)\sin(kx-\omega t)$

$\nabla^2\phi = 0 \Rightarrow \dfrac{d^2 f}{dz^2} - k^2 f = 0 \Rightarrow \underline{Ae^{kz}+Be^{-kz} = f(z)}$

① $z=-H$. $\left.\dfrac{\partial\phi}{\partial z}\right|_{z=-H} = 0$ $\quad \left(\because\ u_z\Big|_{bottom} = 0\right)$.

$\Rightarrow \dfrac{\partial f}{\partial z}\sin(kx-\omega t) = 0 \Rightarrow \left.\dfrac{\partial f}{\partial z}\right|_{z=-H} = 0 \Rightarrow k\left(Ae^{-kH} - Be^{kH}\right) = 0 \Rightarrow \underline{A = Be^{2kH}}$

② $z=0$, $\left.\dfrac{\partial\phi}{\partial z}\right|_{z=0} = \dfrac{\partial\eta}{\partial t}$, Note that $\eta(x,t) = a\cos(kx-\omega t)$ —— ①

$\Rightarrow \dfrac{\partial\eta}{\partial t} = (Ak-Bk)\sin(kx-\omega t) \Rightarrow a\left(-\sin(kx-\omega t)\right)(-\omega) = (Ak-Bk)\sin(kx-\omega t)$

$\Rightarrow \underline{k(A-B) = a\omega} \Rightarrow \underline{A = \dfrac{a\omega}{k}\dfrac{1}{1-e^{-2kH}}}$, $\underline{B = \dfrac{a\omega}{k}\dfrac{e^{-2kH}}{1-e^{-2kH}}}$

$\Rightarrow A = \dfrac{a\omega}{k}\dfrac{e^{kH}}{e^{kH}-e^{-kH}} = \dfrac{a\omega}{k}\dfrac{e^{kH}}{2\sinh kH}$

$B = \dfrac{a\omega}{k}\dfrac{e^{-kH}}{e^{kH}-e^{-kH}} = \dfrac{a\omega}{k}\dfrac{e^{-kH}}{2\sinh kH}$

$\Rightarrow f(z) = e^{kz}\dfrac{a\omega}{k}\dfrac{e^{kH}}{2\sinh kH} + e^{-kz}\dfrac{a\omega}{k}\dfrac{e^{-kH}}{2\sinh kH}$

$= \dfrac{a\omega}{k}\dfrac{e^{k(z+H)}+e^{-k(z+H)}}{\sinh(kH)\cdot 2} = \underline{\dfrac{a\omega}{k}\dfrac{\cosh k(z+H)}{\sinh kH}}$

$\cosh k(z+H)$.

$\Rightarrow \underline{\phi(x,z,t) = \dfrac{a\omega}{k}\dfrac{\cosh k(z+H)}{\sinh kH}\sin(kx-\omega t)}$ —— ②

③ $\left.\dfrac{\partial\phi}{\partial t}\right|_{z=0} = -g\eta$ from ① and ②,

$\Rightarrow -\dfrac{\cancel{a}\omega^2}{k}\dfrac{\cosh k(0+H)}{\sinh kH}\cancel{\cos(kx-\omega t)} = -g\cancel{a}\cancel{\cos(kx-\omega t)}$

$\Rightarrow \omega^2 = gk\dfrac{\sinh kH}{\cosh kH} \Rightarrow \omega = \sqrt{gk\tanh(kH)}$.

* Properties.

$$\begin{cases} V_p = \omega/k = \sqrt{\frac{g}{k}}\sqrt{\tanh(kH)} = \sqrt{gH\left\{\frac{\tanh(kH)}{kH}\right\}} \\ V_g = \partial\omega/\partial k = \frac{1}{2}\sqrt{\frac{g}{k}}\sqrt{\tanh(kH)} = \frac{1}{2}\sqrt{gH\left\{\frac{\tanh(kH)}{kH}\right\}}. \end{cases}$$

$\left(\frac{H}{\lambda} \gg 1 \Rightarrow kH \gg 1\right)$ c_p for deep water $(H \gg \lambda)$	$\left(\frac{H}{\lambda} \ll 1 \Rightarrow kH \ll 1\right)$ c_p for shallow water $(H \ll \lambda)$
$\lim_{kH\to\infty} \tanh kH = 1 \Rightarrow c_p \equiv \sqrt{\frac{g}{k}}$	$\lim_{kH\to 0} \frac{\tanh kH}{kH} = 1 \Rightarrow c_p = \sqrt{gh}$.
$\lambda\uparrow \quad k\downarrow \quad c_p\uparrow$	$H\uparrow \quad c_p\uparrow$ 

* Trajectory of particles.

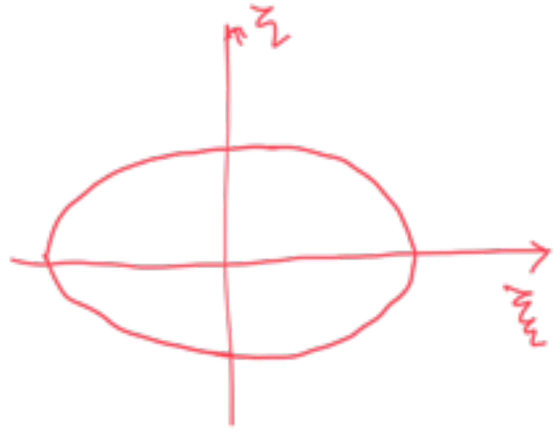
$$\vec{x}_p(t) = x_p(t)\,\vec{a_x} + z_p(t)\,\vec{a_z}$$

$$\left.\begin{aligned} x_p(t) &= x_0 + \xi(t) \\ z_p(t) &= z_0 + \zeta(t) \end{aligned}\right\} \Rightarrow \begin{aligned} \partial x_p/\partial t &= \partial\xi/\partial t = \partial\phi/\partial x = \frac{a\omega}{k}\frac{\cosh k(z+H)}{\sinh kH}\cos(kx-\omega t) \\ \partial z_p/\partial t &= \partial\zeta/\partial t = \partial\phi/\partial z = \frac{a\omega}{k} \end{aligned}$$

$$\begin{cases} \xi(t) = -a\,\frac{\cosh k(z+H)}{\sinh kH}\sin(kx-\omega t) + \text{"}0\text{"} \\ \zeta(t) = a\,\frac{\sinh k(z+H)}{\sinh kH}\cos(kx-\omega t) \end{cases}$$

→ ∵ ξ and ζ means perturbation.

$$\Rightarrow \left\{\frac{\xi(t)}{-a\,\frac{\cosh k(z+H)}{\sinh kH}}\right\}^2 + \left\{\frac{\zeta(t)}{+a\,\frac{\sinh k(z+H)}{\sinh kH}}\right\}^2 = 1$$



Ellipse!

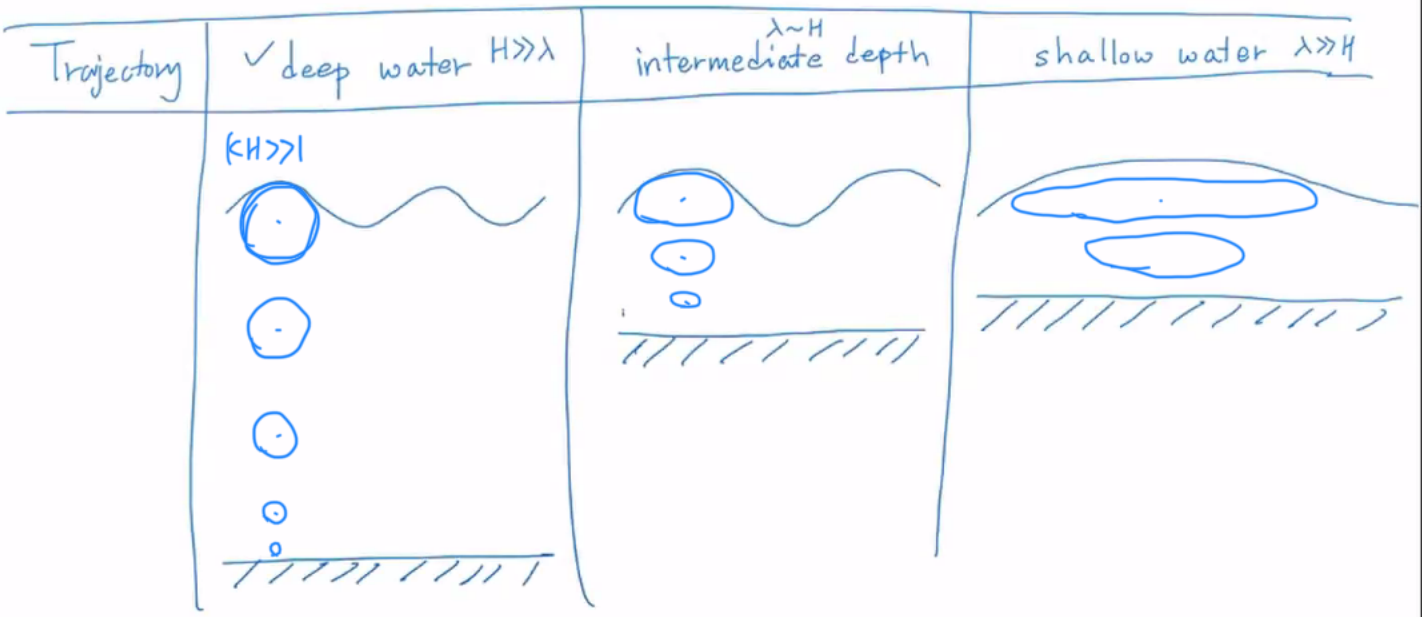
Note $-H < z < 0 \Rightarrow \cosh k(z+H) \propto e^{\square} + e^{-\square} > \sinh k(z+H) \propto e^{\square} - e^{-\square}$

strict!

HW #2 prob 5.

Find the orientation. (direction of rotation)

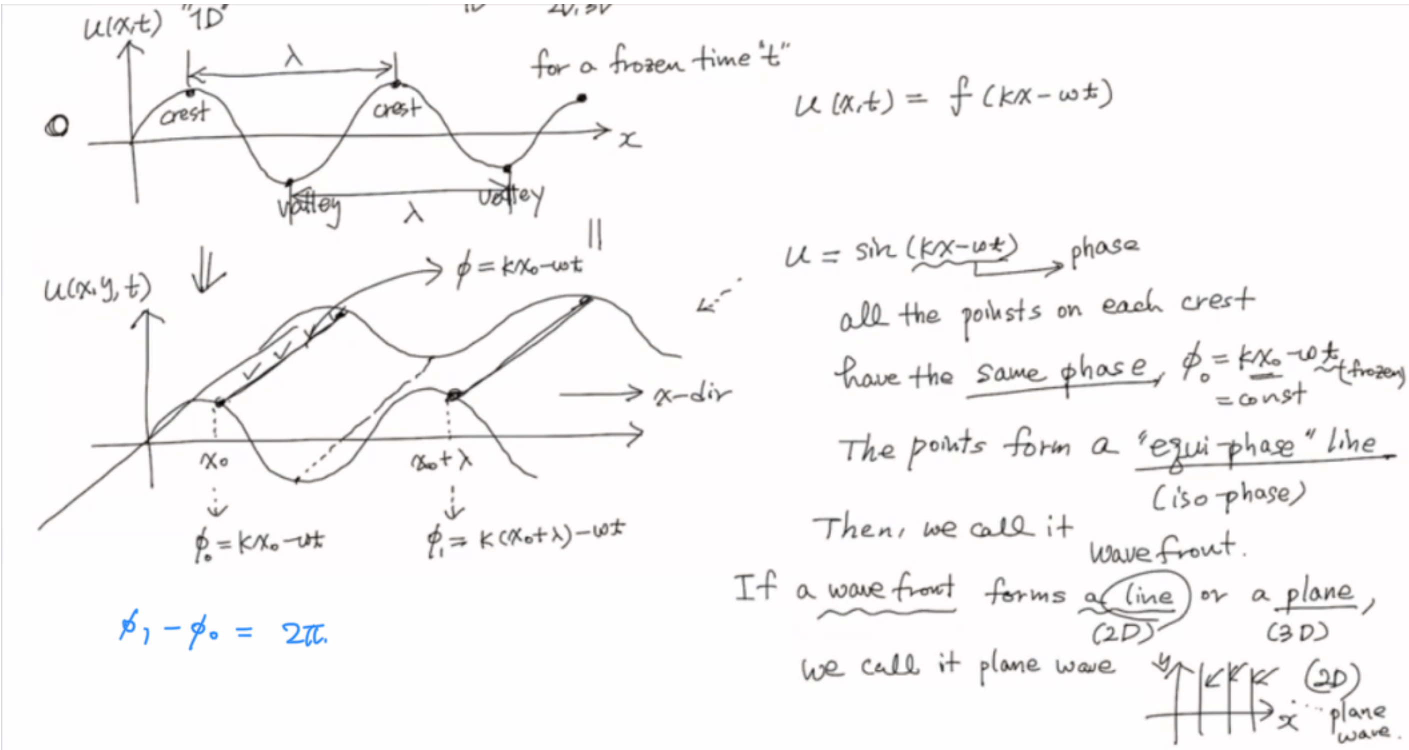
$\Rightarrow$ prove it!



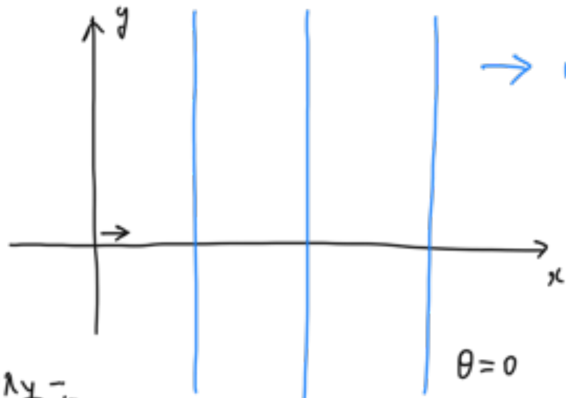
recall $\left[\dfrac{\xi}{a\,\dfrac{\cosh k(z+H)}{\sinh kH}}\right]^2 + \left[\dfrac{\zeta}{a\,\dfrac{\sinh k(z+H)}{\sinh kH}}\right]^2$

Wave Lecture 11 (20211012)

- Wavefronts, Wavenumber, wave packets.



< Top view of 2D wave p.p >



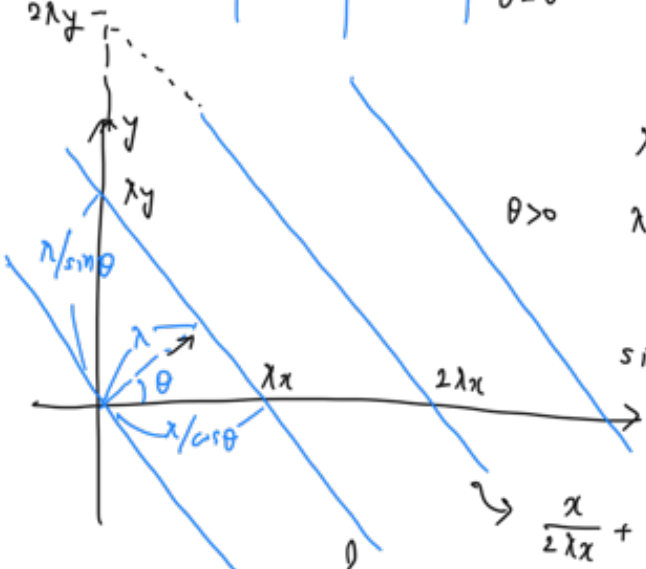
$\rightarrow$ wave fronts (= line)

$\Rightarrow$ plane wave

$\vec{x} = (x, y)$

$\Rightarrow \sin((k, 0) \cdot (x, y) - \omega t)$

$\theta = 0$



$\theta > 0$

$\lambda_x = \frac{\lambda}{\cos\theta} \rightarrow k_x = k\cos\theta$

$\lambda_y = \frac{\lambda}{\sin\theta} \rightarrow k_y = k\sin\theta$

$\sin\{(k_x, k_y)(x, y) - \omega t\}$?

$\frac{x}{2\lambda_x} + \frac{y}{2\lambda_y} = 1 \Rightarrow k_x \cdot x + k_y \cdot y = 4\pi$

2π phase diff

$\frac{x}{\lambda/\cos\theta} + \frac{y}{\lambda/\sin\theta} = 1 \Rightarrow k_x \cdot x + k_y \cdot y = 2\pi$

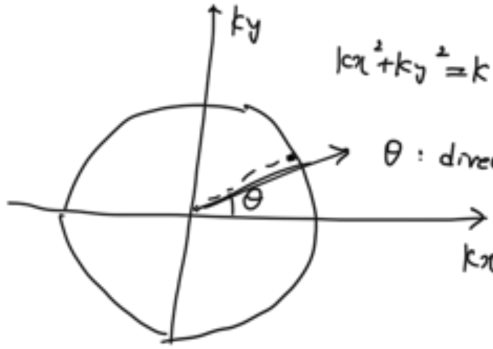
$\Rightarrow$ For all dimensions, $u(\vec{k}\cdot\vec{x} - \omega t)$ (Note : wavefront $\perp \vec{k}$)

E.g.) cylinderical wave

$\vec{x} = (r\cos\theta, r\sin\theta)$, $\vec{k} = (k\cos\theta, k\sin\theta)$

$u(\vec{k}\cdot\vec{x} - \omega t) = u(kr(\cos^2 + \sin^2) - \omega t) = u(kr - \omega t)$

<Wavenumber plane. (2D)>



$k_x^2 + k_y^2 = k^2$

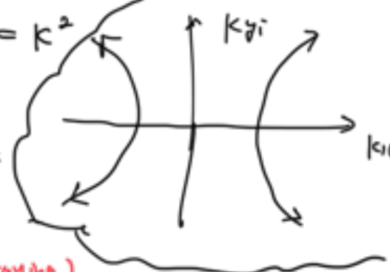
θ : direction of propagation

what if.

Q) $k_x > k$?

$k_x^2 + k_y^2 = k^2$ where $k_y = j k_{y_i}$ $\in \mathbb{R}$.

$\Rightarrow k_x^2 - k_{y_i}^2 = k^2$



Note: $e^{j(\vec{k}\cdot\vec{x} - \omega t)}$

$= e^{jk_x x} \underline{e^{-k_{y_i} y}} e^{-j\omega t}$

Evanescent (Decaying).

<Wave packets> Superposition. Phase velocity. Group velocity.

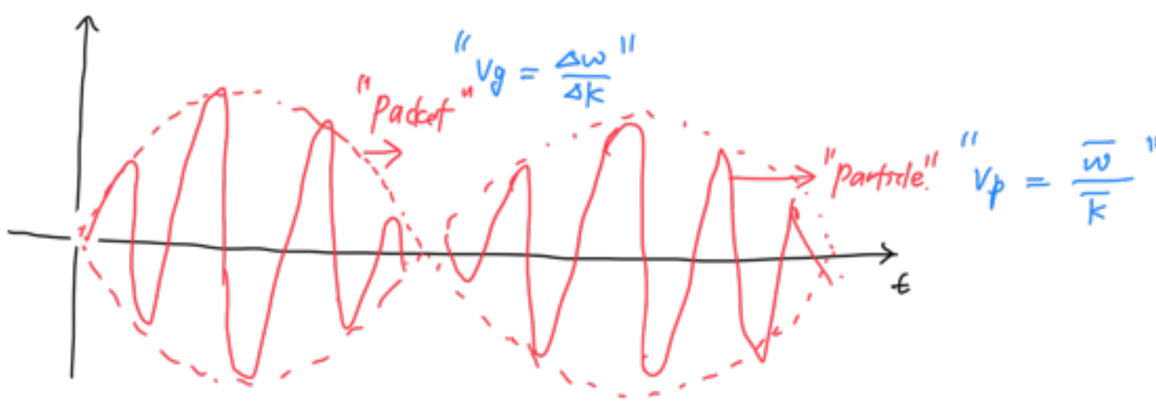
$$\left.\begin{aligned} f_1 &= A\cos(\underbrace{k_1 x - \omega_1 t}_{\alpha}) \\ f_2 &= A\cos(\underbrace{k_2 x - \omega_2 t}_{\beta}) \end{aligned}\right\} \oplus \Rightarrow f_1 + f_2 = A\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$\Rightarrow f_1 + f_2 = 2A\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)$$

$$= 2A \underbrace{\cos(\bar{k}x - \bar{\omega}t)}_{\text{Fastly oscillating}} \underbrace{\cos(\Delta k x - \Delta\omega t)}_{\text{slowly oscillating.}}$$

$\underline{\Delta\omega \ll \bar{\omega}}$ $\because \omega_1 \approx \omega_2$.

$k_1 \approx k_2$



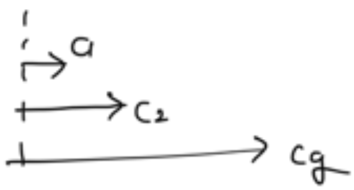
$$c_g = \frac{\Delta\omega}{\Delta k} \rightarrow \lim_{\substack{\Delta k \to 0 \\ \Delta\omega \to 0}} \frac{\Delta\omega}{\Delta k} \Rightarrow \frac{\partial\omega}{\partial k}, \quad c_p = \frac{\omega}{k}$$

Q1. $k_1 \neq k_2$. $w_1 \neq w_2$, what's different?

Q2. $\frac{w_1}{k_1} < \frac{w_2}{k_2}$ and $k_1 < k_2$

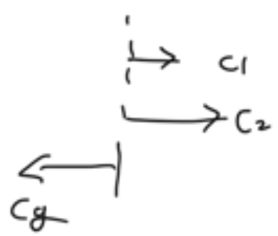
$$c_g = \frac{w_2 - w_1}{k_2 - k_1} > \frac{w_2 - k_1/k_2 \cdot w_2}{k_2 - k_1} = \frac{w_2}{k_2} = c_2$$

$\Rightarrow$ $c_g > c_2 > c_1$ (Group velocity is the fastest among c_1 & c_2)



Q3. $\frac{w_1}{k_1} > \frac{w_2}{k_2}$ and $k_1 < k_2$ and $w_1 > w_2$

$c_g = \frac{w_2 - w_1}{k_2 - k_1} < 0$ (wave packets move in the opposite direction)



Wave Lecture 12 (20211014)

Moving medium wave equation

$(\rho = \rho_0 + \rho' ,\ u = u_0 + u' ,\ p = p_0 + p')$

① cont : $\partial \rho / \partial t + \nabla \cdot (\rho \vec{u}) = 0 \Rightarrow \partial \rho' / \partial t + \nabla (\rho_0 u' + \rho' u_0) = 0$

$$\Rightarrow \underbrace{\partial \rho' / \partial t + \rho_0 \nabla u'}_{\text{Stationary } (u_0 = 0)} + u_0 \cdot \nabla \rho' = 0.$$

Euler.

② $\rho \left(\partial u / \partial t + u \cdot \nabla u \right) = -\nabla p \Rightarrow (\rho_0 + \rho') \left(\partial u' / \partial t + \underbrace{(u_0 + u') \nabla (\cancel{u_0} + u')}_{H.O.T \Rightarrow 0} \right) = -\nabla p'$

$$\Rightarrow \underbrace{\rho_0 \left(\partial u' / \partial t\right.}_{\text{Stationary } (u_0 = 0)} \left. + u_0 \cdot \nabla u' \right) = -\nabla p'$$

③ Isentropic $(s = 0)$ state.

$p' = c^2 \rho' = (\sqrt{\gamma R T})^2 \rho'$

$$\boxed{\begin{aligned} &\text{let } \partial/\partial t + u \cdot \nabla = \frac{D}{Dt} \\ &\partial/\partial t + u_0 \cdot \nabla = \frac{D_0}{D_0 t} \end{aligned}}$$

$\Rightarrow \frac{D_0}{D_0 t} \rho' + \rho_0 \nabla \cdot u' = 0$ — ①

$\rho_0 \cdot \frac{D_0}{Dt} u' = -\nabla p'$ — ②

$\Rightarrow \frac{D_0}{D_0 t}$ ① $- \nabla \cdot$ ②

$$\Rightarrow \nabla^2 p' - \frac{1}{c^2} \frac{D_0^2}{D_0 t^2} p' = 0$$

$$\nabla^2 p' - \frac{1}{c^2} \left(\frac{D_0^2}{D_0 t^2} p' \right) = 0 \Rightarrow \nabla^2 p' - \frac{1}{c^2} \left(\underbrace{\partial^2 / \partial t^2}_{①} + \underbrace{2 \vec{u}_0 \cdot \nabla\, \partial / \partial t}_{②} + \underbrace{(\vec{u}_0 \cdot \nabla)^2}_{③} \right) p' = 0 .$$

$p'(x,t) = \tilde{p}(\vec{x})\, e^{-j\omega t}$

Note that $\vec{x}$ is vector $(\vec{x} = (x, y))$

① $\frac{\partial^2 p'}{\partial t^2} = (-j\omega)^2 \tilde{p}(\vec{x})\, e^{-j\omega t} = -\omega^2 \tilde{p}(\vec{x})\, e^{-j\omega t}$

② $2\vec{u}_0 \cdot \nabla\, \partial/\partial t\, p' = 2\vec{u}_0 \cdot \nabla (-j\omega)\, \tilde{p}(x) e^{-j\omega t} = \underline{-2j\omega\, u_\infty \frac{\partial \tilde{p}}{\partial x} e^{-j\omega t}}$

$\underline{-2j\omega\, (u_x, u_y)\cdot\left(\frac{\partial \tilde{p}}{\partial x}, \frac{\partial \tilde{p}}{\partial y}\right) e^{-j\omega t}}$.

(for u_x, u_y, $u_x \neq u_\infty$)

$\because$ Assume $\vec{u}_0 = (u_\infty, 0)$ in $\mathbb{R}^2$

$\Rightarrow \vec{u}_0 \cdot \nabla = (u_\infty, 0)(\partial/\partial x, \partial/\partial y) = u_\infty\, \partial/\partial x$.

③ $u_0 \cdot \nabla = u_\infty \frac{\partial}{\partial x} \Rightarrow \left(u_\infty \frac{\partial}{\partial x}\right)^2 p' = \underline{u_\infty^2 \frac{\partial^2 \tilde{p}'}{\partial x^2}}$

$u_0 \cdot \nabla = (u_x, u_y)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} \Rightarrow$ square $\Rightarrow \underline{u_x^2 \frac{\partial^2 \tilde{p}}{\partial x^2} + 2u_x u_y \frac{\partial^2 \tilde{p}}{\partial x \partial y} + 2u_y^2 \frac{\partial^2 \tilde{p}}{\partial y^2}}$.

$$\Rightarrow \left[\nabla^2 \tilde{p}(\vec{x}) - \frac{1}{c^2}\left\{-\omega^2 \tilde{p}(\vec{x}) - 2j\omega\, u_\infty \frac{\partial \tilde{p}}{\partial x} + u_\infty^2 \frac{\partial^2 \tilde{p}'}{\partial x^2}\right\}\right] e^{-j\omega t} = 0$$

$$\Rightarrow \frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 \tilde{p} + 2j\left(\frac{\omega}{c}\right)\left(\frac{u_\infty}{c}\right)\frac{\partial \tilde{p}}{\partial x} - \left(\frac{u_\infty}{c}\right)^2 \frac{\partial^2 \tilde{p}}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + k^2 \tilde{p} + 2jkM \frac{\partial \tilde{p}}{\partial x} - M^2 \frac{\partial^2 \tilde{p}}{\partial x^2} = 0$$

$\left(\begin{array}{ll} M = \frac{u_\infty}{c} : \text{Mach \#} \\ k = \frac{\omega}{c} : \text{wave \#} \end{array}\right)$ $M=1$: Sonic, $M>1$: super Sonic

$$\Rightarrow \boxed{(1-M^2)\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + 2jkM\frac{\partial \tilde{p}}{\partial x} + k^2 \tilde{p} = 0}$$

if $M=0$, $\nabla^2 \tilde{p} + k^2 \tilde{p} = 0$

$\Rightarrow$ Helmholtz Eq.

< In 1-Dimension >

$$\Rightarrow (1-M^2)\frac{\partial^2 \tilde{p}}{\partial x^2} + 2jkM \frac{\partial \tilde{p}}{\partial x} + k^2 \tilde{p} = 0$$

$\underline{\text{set } \tilde{p} = e^{jk_M x}}$

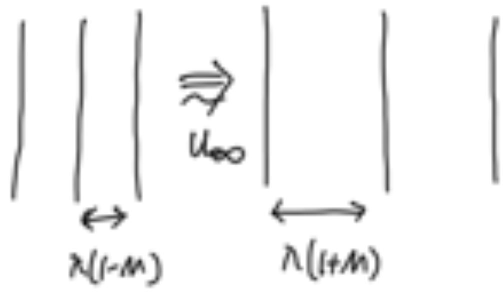
(Note : $k \neq k_M$)

$$\Rightarrow (1-M^2)(-k_M^2) + 2jk\cdot M(jk_M) + k^2 = 0$$

$$\Rightarrow \underline{(1-M^2)k_M^2 + 2k k_M \cdot M - k^2 = 0}. \Rightarrow k_M = \frac{-kM \pm \sqrt{k^2M^2 + k^2(1-M^2)}}{1-M^2}$$

$$\Rightarrow k_M = \frac{-kM \pm k}{(1-M)(1+M)} \left\langle \begin{array}{l} \frac{k}{1+M} \\ \frac{-k}{1-M}. \end{array}\right.$$

$$\Rightarrow \tilde{p}(x) = \begin{cases} e^{j\frac{k}{1+M}x} & : \text{right-going} \quad \lambda_r = (1+M)\lambda \\ e^{-j\frac{k}{1-M}x} & : \text{left-going} \quad \lambda_l = (1-M)\lambda. \end{cases}$$



$\Rightarrow$ Modulation of wavelength due to the flow.

< Prandtl – Glauert Transformation (P–G trans) >

$$(1-M^2)\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + 2jkM\frac{\partial \tilde{p}}{\partial x} + k^2\tilde{p} = 0$$

$$\xi = \frac{x}{\sqrt{1-M^2}} \quad \text{and} \quad \eta = y \quad \text{and} \quad \psi = \tilde{p}\, e^{-j\alpha\xi}$$

? 나중에 Calibration.

$\hookrightarrow$ $(1-M^2)$ 상쇄시키기 위함

$$\Rightarrow \cancel{(1-M^2)}\frac{1}{\cancel{1-M^2}}\frac{\partial^2}{\partial \xi^2}\left(\psi e^{j\alpha\xi}\right) + \frac{\partial^2}{\partial \eta^2}\left(\psi e^{j\alpha\xi}\right) + 2jkM\frac{\partial}{\partial x}\left(\psi e^{j\alpha\xi}\right) + k^2\psi e^{j\alpha\xi} = 0$$

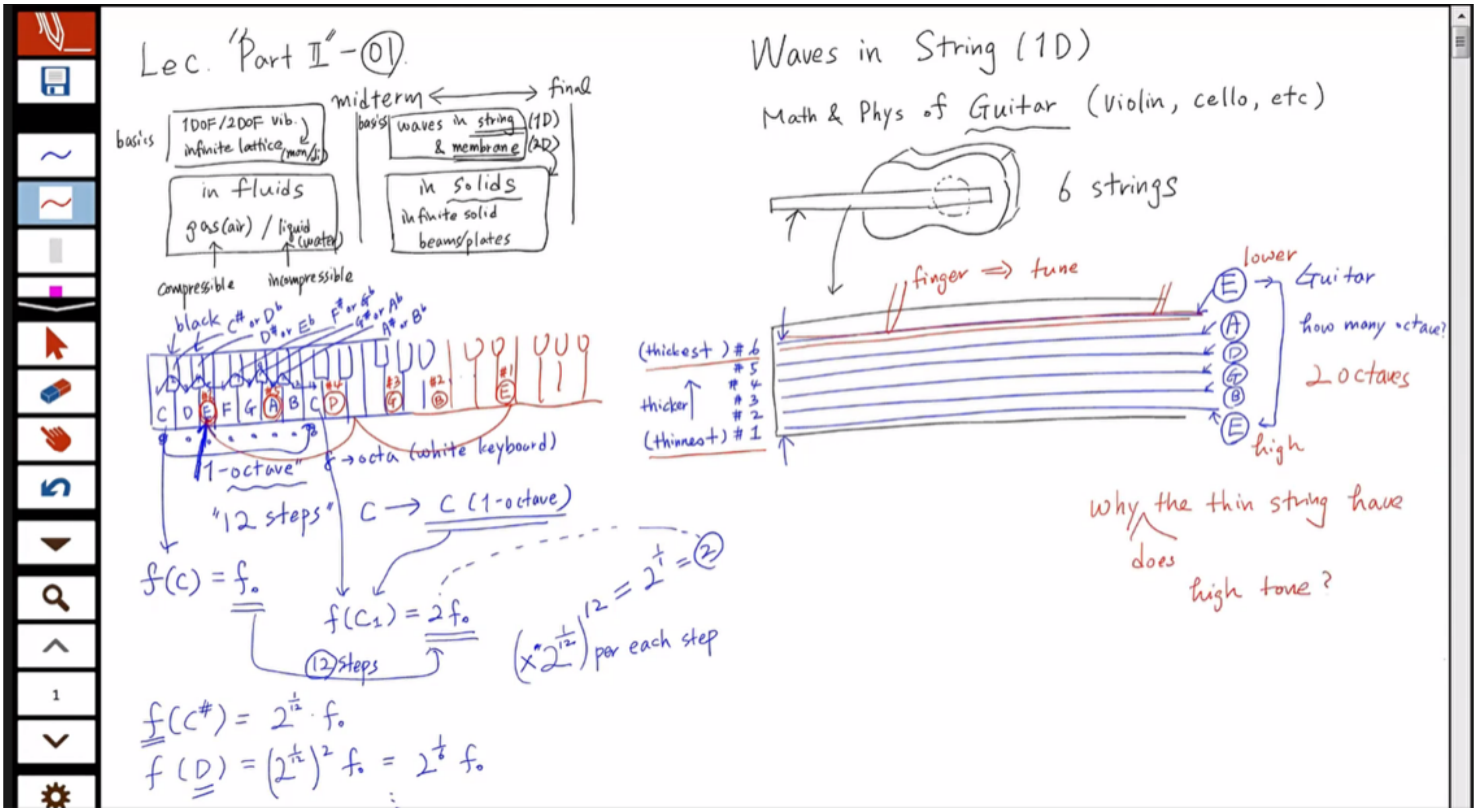
$$\Rightarrow \frac{\partial^2\psi}{\partial\xi^2}\cancel{e^{j\alpha\xi}} + 2(j\alpha)\frac{\partial\psi}{\partial\xi}\cancel{e^{j\alpha\xi}} + (j\alpha)^2\psi\cancel{e^{j\alpha\xi}} + \frac{\partial^2\psi}{\partial\eta^2}\cancel{e^{j\alpha\xi}} + \frac{2jkM}{\sqrt{1-M^2}}\left(\frac{\partial\psi}{\partial\xi} + j\alpha\psi\right)\cancel{e^{j\alpha\xi}} + k^2\psi\cancel{e^{j\alpha\xi}} = 0$$

$$\Rightarrow \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} + 2j\left(\alpha + \frac{kM}{\sqrt{1-M^2}}\right)\frac{\partial\psi}{\partial\xi} + \psi\left\{k^2 - \alpha^2 - \frac{2kM\alpha}{\sqrt{1-M^2}}\right\}$$

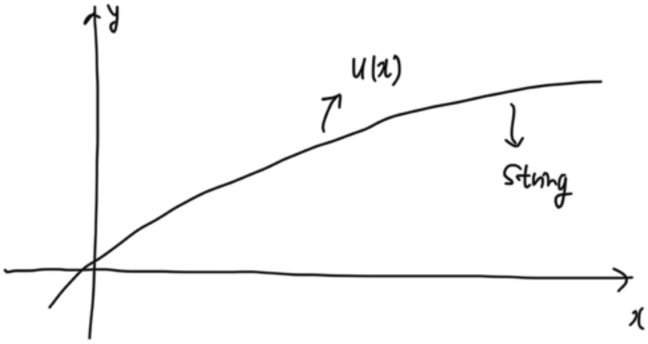
\# set $\alpha = \frac{-kM}{\sqrt{1-M^2}}$ to eliminate $\partial\psi/\partial\xi$ term!

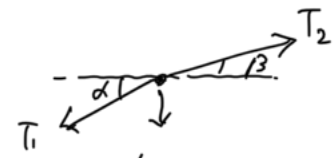
$$\Rightarrow \nabla^2\psi + \psi\left\{k^2 - \frac{k^2M^2}{1-M^2} + \frac{2k^2M^2}{1-M^2}\right\} \Rightarrow \boxed{\nabla^2\psi + \psi\left\{\frac{k^2}{1-M^2}\right\} = 0}$$

Wave Lecture 13 (20211026)



· Waves in String





$dm = \rho\, dx$

1) x : $-T_1\cos\alpha + T_2\cos\beta = 0 \Rightarrow T_1 \approx T_2 = T.$

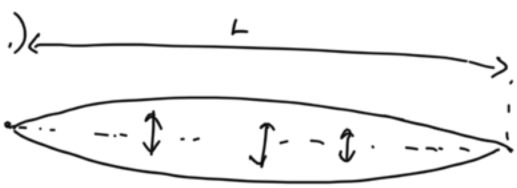
2) y : $T_2\sin\beta - T_1\sin\alpha = dm\,\ddot{u} \Rightarrow T(\tan\beta - \tan\alpha) = dm\,\ddot{u} = \rho\,\Delta x \cdot \ddot{u}$

$$\Rightarrow \tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x = x_0 + \Delta x} \quad \text{and} \quad \tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{x = x_0}$$

$$\Rightarrow T\left\{ \left.\frac{\partial u}{\partial x}\right|_{x = x_0+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x = x_0} \right\} = T\left\{ \cancel{\left.\frac{\partial u}{\partial x}\right|_{x_0}} + \frac{\partial}{\partial x}\left.\frac{\partial u}{\partial x}\right|_{x_0} \cdot \Delta x + \cancel{H.O.T}^{\,0} - \cancel{\left.\frac{\partial u}{\partial x}\right|_{x_0}} \right\}$$

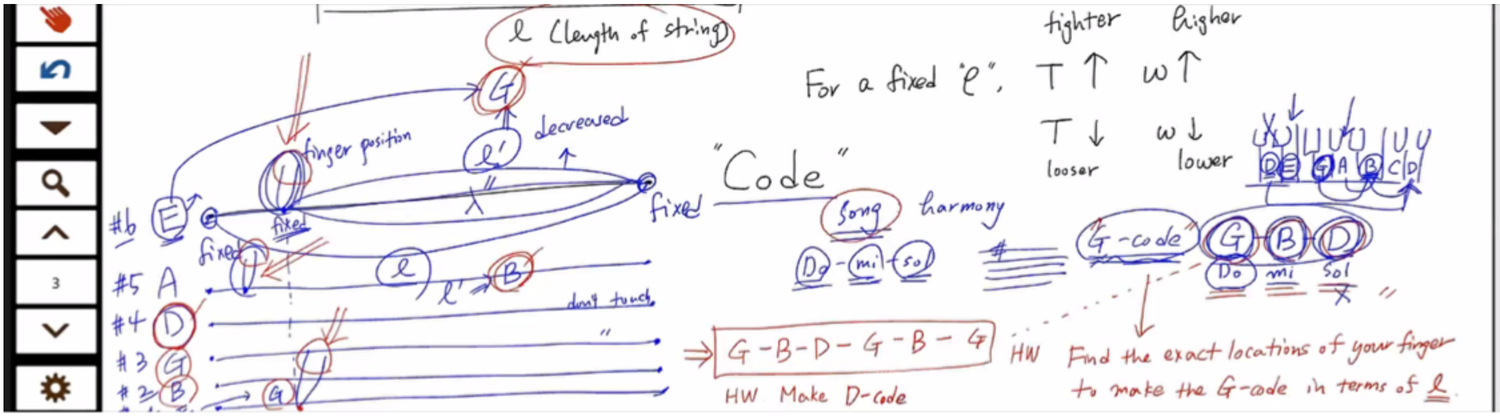
$$\Rightarrow T\,\frac{\partial^2 u}{\partial x^2}\,\cancel{\Delta x} = \rho \cdot \cancel{\Delta x} \cdot \frac{\partial^2 u}{\partial t^2} \Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{C_s^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \left(C_s = \sqrt{T/\rho} \right)}$$

E.g.)



$\lambda = 2L \rightarrow k = \frac{2\pi}{\lambda} = \frac{\pi}{L}$

$\omega = kc = \frac{\pi}{L} \cdot \sqrt{\frac{T}{\rho}}$



D : D − F$^{\#}$ − A

A : A − C$^{\#}$ − E

주로. GDC c·de 함께 사용

Next Class : String G.E using Lagrangian Mechanics. (For continuum system)

Not lattice!

Wave Lecture 14 (20211028)

Waves in Strings (Lagrangian)

- $T = \int_{x_1}^{x_2} \frac{1}{2}(\rho\, dx)\left(\frac{du}{dt}\right)^2$, V = Work done in "stretching" the string away from equilibrium.

Tension : k $\Rightarrow \Delta V = k \cdot \Delta l \Rightarrow \Delta l = \sqrt{(dx)^2 + (du)^2} - dx$

$$\Rightarrow \Delta l = dx\left\{\sqrt{1+\left(\frac{du}{dx}\right)^2} - 1\right\} \approx dx\left\{1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2 - 1\right\} = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx \quad \left(\because \begin{array}{l} u \ll 1 \\ u_x, u_t \ll 1 \end{array}\right)$$

$$V = \int_{x_1}^{x_2} \delta V = \int_{x_1}^{x_2} k\, \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx$$

$$\mathcal{L} = T - V = \int_{x_1}^{x_2} \left\{\frac{1}{2}\rho\,(\partial u/\partial t)^2 - \frac{1}{2}k\,(\partial u/\partial x)^2\right\} dx$$

$L = L(t, x, u, u_t, u_x)$

$\Rightarrow$ Euler-Lagrange Eq. can't be used.

$\Rightarrow$ Generalized Euler-Lagrange Eq. : $\partial L/\partial u - \partial/\partial t\,(\partial L/\partial u_t) - \partial/\partial x\,(\partial L/\partial u_x) = 0$

① $\partial L/\partial u = 0$

② $\partial L/\partial u_t = \rho\, \partial u/\partial t$

③ $\partial L/\partial u_x = -k\, \partial u/\partial x$

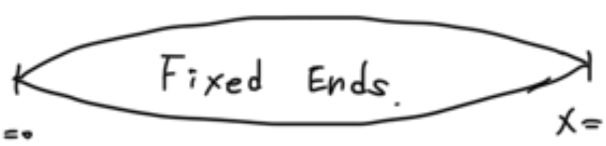
$$\Rightarrow -\rho\, \partial^2 u/\partial t^2 + k\, \partial^2 u/\partial x^2 = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{k}{\rho}\frac{\partial^2 u}{\partial x^2} = 0$$

Wave Eq. for string.

- Solving Equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \rightarrow u(x,t) = X(x)T(t) \Rightarrow \ddot{T}X - c^2 T\ddot{X} = 0 \Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\mu^2.$$



$x=0$ $x=L$

$u(0) = 0$ $u(L) = 0$

$X(0) = 0$ $X(L) = 0$

$X(x) = A\cos\mu x + B\sin\mu x.$

$A = 0,\ \mu L = m\pi. \Rightarrow \mu_m = \frac{m\pi}{L}$ $(m = 1, 2, \cdots).$

$$X(x) = \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi}{L}x\right)$$

$$T(t) = A'\cos(c\mu t) + B'\sin(c\mu t) \Rightarrow u(x,t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right)\left[C_m \cos\left(c\cdot\frac{m\pi}{L}t\right) + D_m \sin\left(c\frac{m\pi}{L}t\right)\right]$$

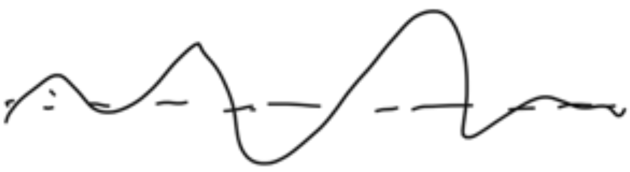
C_m, D_m ?

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \begin{aligned} &\to \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right) C_m = f(x) \quad \text{———} \quad 1. \\ &\to \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right)\cdot c\cdot\frac{m\pi}{L}\cdot D_m = g(x) \quad \text{———} \quad 2. \end{aligned}$$

① $\displaystyle\int_0^{2L/m} \sin\left(\frac{m\pi x}{L}\right)\cdot\sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right)\cdot C_m\, dx = \int_0^{2L/m} \sin\left(\frac{m\pi x}{L}\right)\cdot f(x)\, dx$

② In the same way as ①.

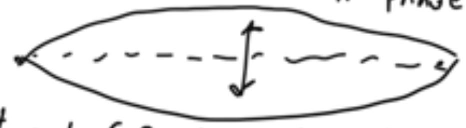
physics : $u(x,t) = \displaystyle\sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right)\left(\text{Temporal terms}\right)$



$= \sum ① + ② + ③ + \cdots$

① $m=1$

In - phase



1st mode / fundamental mode.

② $m=2$

Node / Nodal point.

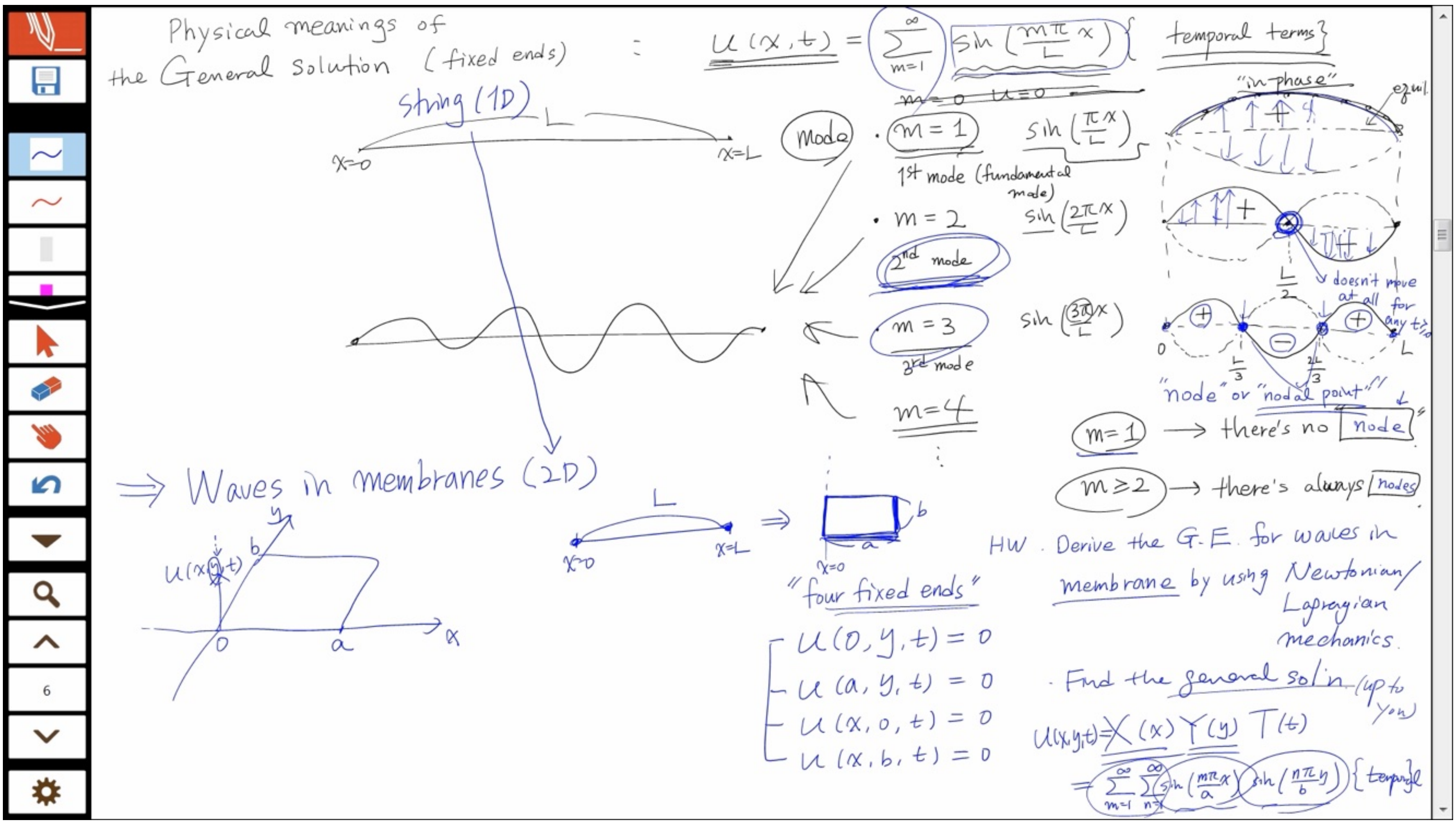
2nd mode.

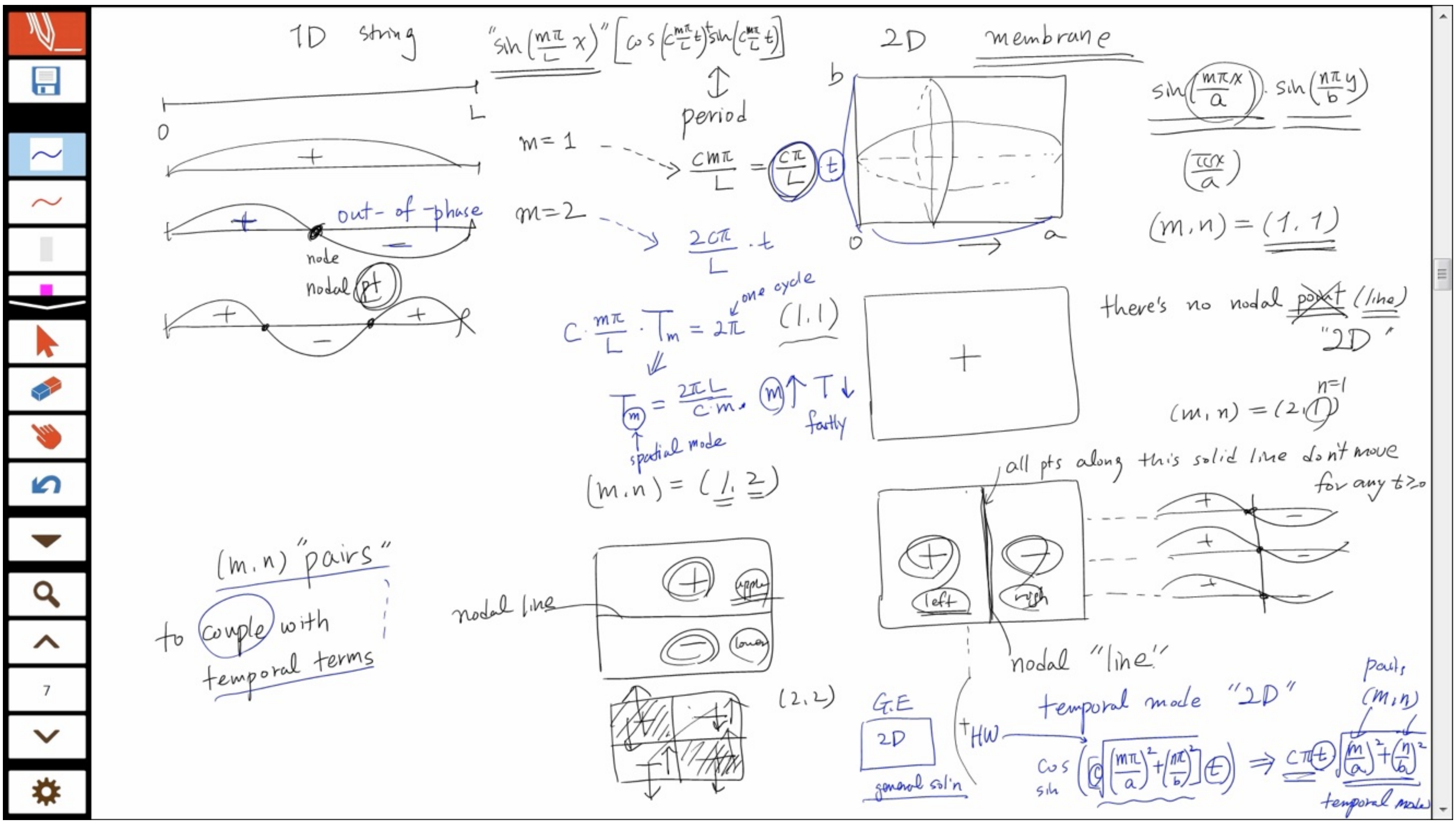
③ $m=3$

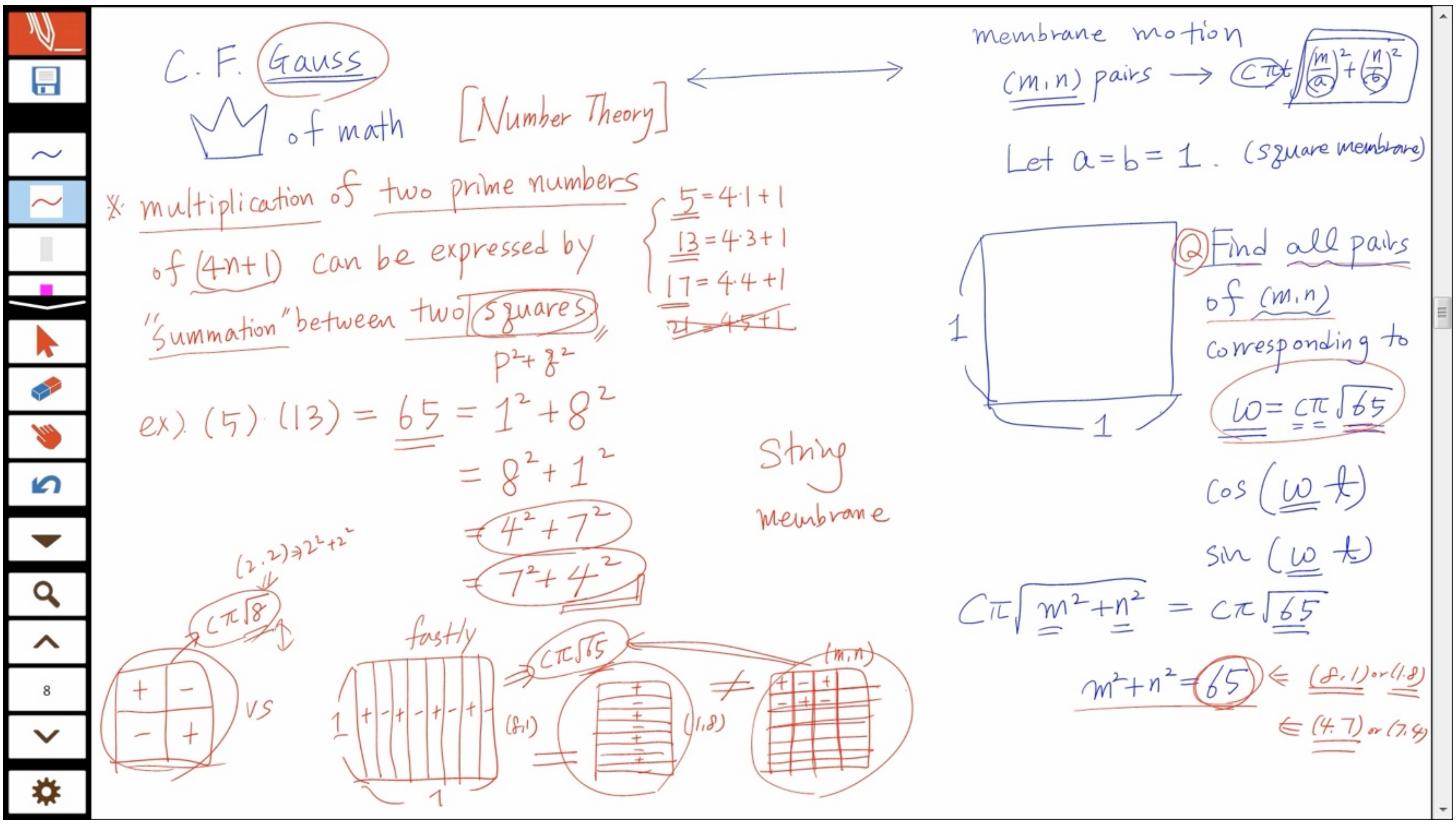
$m=1 \to$ no node

$m \geq 2 \to$ always node.

Wave Lecture 15 (20211102)







$$u_{12} \qquad\qquad u_{21}$$

① $\sin(\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(\pi y)$

$$2\sin\pi x\,\sin\pi y\cos\pi y + 2\sin\pi x\cos\pi x\sin\pi y$$

$$= 2\sin\pi x\sin\pi y\,\underbrace{(\cos\pi x + \cos\pi y)}_{=0} = 0$$

② $\frac{1}{2}\sin(\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(\pi y)$

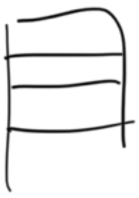
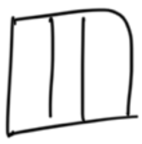
 

$$\sin\pi x\,\sin\pi y\cos\pi y + 2\sin\pi x\cos\pi x\,\sin\pi y$$

$$= \sin\pi x\sin\pi y\,\underbrace{(\cos\pi y + 2\cos\pi x)}_{=0} = 0$$

$$u_{13} \qquad\qquad u_{31}$$

③ $\sin(\pi x)\sin(3\pi y) + \sin(3\pi x)\sin(\pi y)$

$$\sin\pi x\,(3\sin\pi y - 4\sin^3\pi y)$$

$$+ \sin\pi y\,(3\sin\pi x - 4\sin^3\pi x)$$

$$= 3\sin\pi x\sin\pi y\,\underbrace{(6 - 4\sin^2\pi x - 4\sin^2\pi y)}_{=0} = 0$$

① $\cos\pi x + \cos\pi y = 0$

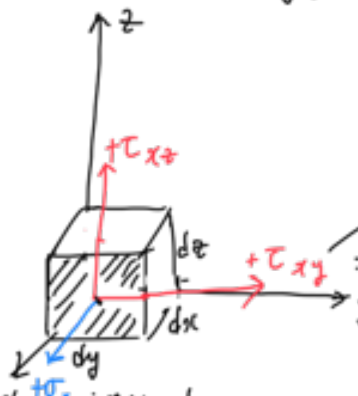
$$\cos\pi x + \frac{1}{2}\cos\pi y = 0$$

Wave Lecture 16 (20211104)

$\boxed{\text{Elastodynamics}}$: Dynamic theory of elasticity.

· Basic background.
1) stress
2) strain
3) Displacement
4) modulus — E (Young's), G. (shear)
5) Poisson ratio ν
6) Hooke's law · $\sigma_x = E\varepsilon_x$
7) Cauchy, Lané

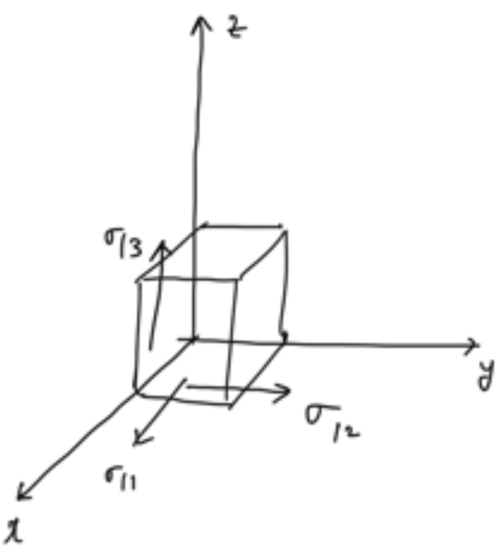
① Stress : contact force / unit area of an infinitesimal element.
≠ body force (e.g. gravity).



$+\tau_{xy}$: shear stress → y component of shear stress on surface perpendicular to x-axis

$+\sigma_x$: normal stress → on surface perpendicular to x-axis.

Normal : $\pm\sigma_x$, $\pm\sigma_y$, $\pm\sigma_z$
shear : $\pm\tau_{xy}$ $\pm\tau_{xz}$ $\pm\tau_{yz}$

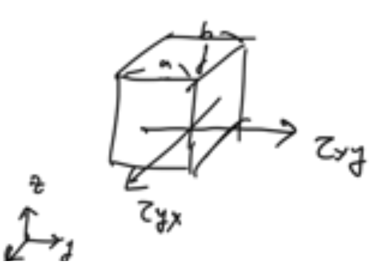


$$\begin{pmatrix} \sigma_x = \sigma_{11} & \sigma_y = \sigma_{22} & \sigma_z = \sigma_{33} \\ \tau_{xy} = \sigma_{12} & \tau_{yz} = \sigma_{23} & \sigma_{zx} = \sigma_{31} \\ \tau_{xz} = \sigma_{13} & \tau_{yx} = \sigma_{21} & \sigma_{zy} = \sigma_{32} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \rightarrow \text{stress tensor.}$$

σ_{ij} = Normal $(i = j)$
Shear $(i \neq j)$

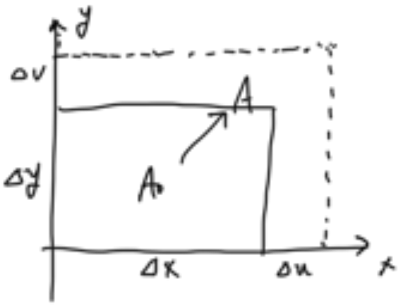
HW) prove $\tau_{xy} = \tau_{yx}$ (state condition for equality).



$T_z = I_z \cdot \alpha = 0 = a\,\tau_{xy} - b\,\tau_{yx}$

$\Rightarrow \tau_{xy} = \tau_{yx}$ $(\underbrace{a = b}_{\text{condition}})$

② Dilatation (2D)



$A_0 = \Delta x \Delta y$

$A = (\Delta x + \Delta u)(\Delta y + \Delta v) \approx \Delta x \Delta y + \Delta x \Delta v + \Delta y \Delta u$

$\Rightarrow A - A_0 = \Delta x \Delta v + \Delta y \Delta u$

Relative change ; $\dfrac{A - A_0}{A_0} = \dfrac{\Delta u}{\Delta x} + \dfrac{\Delta v}{\Delta y}$

$$\varepsilon = \lim_{\substack{\Delta x \Delta y \to 0 \\ \Delta u \Delta v}} R = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot (u, v) = \nabla \cdot \vec{d}$$

Dilatation (scalar)

displacement vector $\vec{d}$

Divergence of displacement.

Note that $\partial u/\partial x = \varepsilon_x$ and $\partial v/\partial y = \varepsilon_y$ (strain)

$\partial w/\partial z = \varepsilon_z$ $\Rightarrow$ $\varepsilon_{2D} = \varepsilon_x + \varepsilon_y$ $\rightarrow$ $\varepsilon_{3D} = \varepsilon_x + \varepsilon_y + \varepsilon_z$

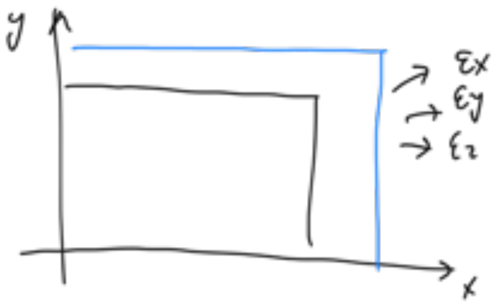
Strain : Tensor

$$\begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{pmatrix}$$

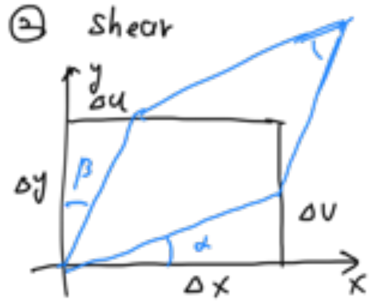
③ Strain - Displacement Relation.

$$\vec{d} = (u, v, w) = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} = (u_1, u_2, u_3)$$

① Normal



② Shear



$$\gamma_{xy} = \alpha + \beta \simeq \tan\alpha + \tan\beta$$
$$= \lim \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Suppose $(\quad)_{,i} = \frac{\partial}{\partial x_i}(\quad)$

$$\Rightarrow \gamma_{xy} = u_{2,1} + u_{1,2}$$
$$\gamma_{12} = u_{1,2} + u_{2,1}$$
$$\gamma_{21} = u_{2,1} + u_{1,2}$$

$\Rightarrow \gamma_{12} = \gamma_{21}$

$(\gamma_{xy} = \gamma_{yx})$

French mathematician

Cauchy) Let $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $\rightarrow$ $\varepsilon_{11} = \frac{1}{2}\cdot 2u_{1,1} = \varepsilon_x$

$\gamma_{12} = u_{1,2} + u_{2,1} = 2\varepsilon_{12}$

$$\begin{pmatrix} \varepsilon_x & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \varepsilon_y & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \varepsilon_z \end{pmatrix} \Longrightarrow \begin{pmatrix} \varepsilon_{11} & 2\varepsilon_{12} & 2\varepsilon_{13} \\ 2\varepsilon_{21} & \varepsilon_{22} & 2\varepsilon_{23} \\ 2\varepsilon_{31} & 2\varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$ $\rightsquigarrow$ Strain tensor.

Summary :

6 stresses
6 strains
3 displacements

15 unknown

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ } 6 Equations.

9 More equations needed!

Wave Lecture 17 (20211109)

~ Previous Class.

① stress → 3 normal, 3 shear (6) → ⑥ unknowns.

② dilation → 3 " 3 " → ⑥ "

③ relation ($\sigma \sim \varepsilon$) → 6 relations

$\nabla \cdot \vec{d}$ = dilation

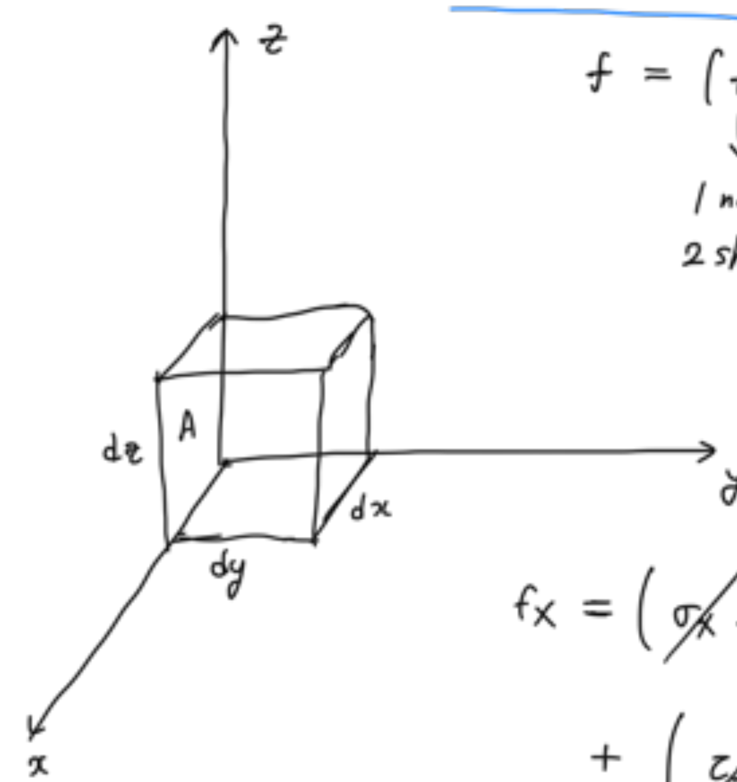
③ unknowns

15 unknowns!

Equation of motion

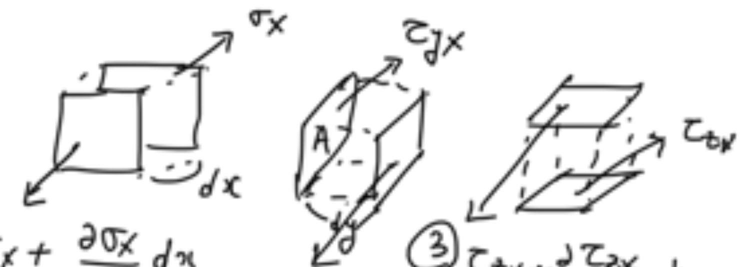
$$\vec{f} = d\vec{F} = dm \cdot \vec{a} = \rho \, dx dy dz \cdot \frac{\partial^2 \vec{d}}{\partial t^2}.$$

($\vec{d}$: displacement vector).



$f = (f_x, f_y, f_z)$

1 normal, 2 shear " "



① $\sigma_x + \frac{\partial \sigma_x}{\partial x} dx$

② $\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy$

↳ A면 기준 가정.

③ $\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz$.

From ①, ②, ③, we get.

$$f_x = \left(\cancel{\sigma_x} + \frac{\partial \sigma_x}{\partial x} dx - \cancel{\sigma_x}\right) dy dz + \left(\cancel{\tau_{yx}} + \frac{\partial \tau_{yx}}{\partial y} dy - \cancel{\tau_{yx}}\right) dx dz$$
$$+ \left(\cancel{\tau_{zx}} + \frac{\partial \tau_{zx}}{\partial z} dz - \cancel{\tau_{zx}}\right) dx dy$$

$$\Rightarrow f_x = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx dy dz.$$

i) x direction.

$$f_x = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \cancel{dx dy dz} = \rho \cancel{dx dy dz} \cdot \partial^2 u / \partial t^2 \qquad (\because \vec{d} = u, v, w)$$

$u = u_1$, $v = u_2$, $w = u_3$

$$\Rightarrow \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} = \sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} = \rho \ddot{u}_1$$

ii) y direction $\Rightarrow$ $\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} = \rho \ddot{u}_2$

iii) z direction $\Rightarrow$ $\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = \rho \ddot{u}_3$

$$\boxed{\sigma_{ji,j} = \rho \ddot{u}_i}$$

for $i = 1, 2, 3$

j : summation index

HW) Derive (ii) and (iii). for excercise!

Cauchy : $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $i,j = 1,2,3$. 6 Equations.

→ strain – displacement

Navier : $\sigma_{ji,j} = \rho \ddot{u}_i$ for $i = 1,2,3$. 3 Equations.

→ stress – displacement.

Lame' : strain – stress via Hooke's law $(\sigma_x = E\varepsilon_x)$ <1D> 6 Equations.

Note : $\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\varepsilon_x(1-\nu) + (\varepsilon_y + \varepsilon_z)\nu\right]$ <3D>

$$\Rightarrow \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\varepsilon_x(1-2\nu) + \underbrace{(\varepsilon_x + \varepsilon_y + \varepsilon_z)}_{=\varepsilon}\nu\right]$$

$$= \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon}_{\text{independent of direction.}} + \frac{E}{1+\nu}\varepsilon_x \quad \text{and} \quad \tau_{xy} = \overset{G.}{\overset{\parallel}{\frac{E}{2(1+\nu)}}}\gamma_{xy}$$

Can we unify these two?

$\varepsilon_x = \varepsilon_{11}$ Let $\frac{E}{2(1+\nu)} = \mu$ and $\frac{E\nu}{(1+\nu)(1-2\nu)} = \lambda$.

$\gamma_{xy} = 2\varepsilon_{12}$

Lame's 2nd const — Lame's 1st const

Rewriting equations)

$$\left.\begin{array}{l|l} \sigma_{11} = \lambda\varepsilon + 2\mu\varepsilon_{11} & \overset{\tau_{xy}}{\overset{\parallel}{\sigma_{12}}} = \mu\,\overset{\gamma_{xy}}{\overset{\parallel}{2\varepsilon_{12}}} \\ \sigma_{22} = \lambda\varepsilon + 2\mu\varepsilon_{22} & \sigma_{23} = \mu\,2\varepsilon_{23} \\ \sigma_{33} = \lambda\varepsilon + 2\mu\varepsilon_{33} & \sigma_{31} = \mu\,2\varepsilon_{31} \end{array}\right\} \Rightarrow \sigma_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda\varepsilon$$

where δ_{ij} → Kronecker delta. func.

$$\delta_{ij} = \begin{pmatrix} 0 & i \neq j \\ 1 & i = j \end{pmatrix}.$$

Since $\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{kk}$ (ε_{kk} : Einstein summation tensor)

$$\Rightarrow \boxed{\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}}$$

Lame' Einstein Kronecker Cauchy.

Conclusion

① $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $i,j = 1,2,3$ 6 strain – displacement

② $\sigma_{ji,j} = \rho\ddot{u}_i$ $i = 1,2,3$ 3 stress – displacement

③

(3) $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$ $\quad i,j = 1,2,3$ $\quad$ 6 $\quad$ stress - strain

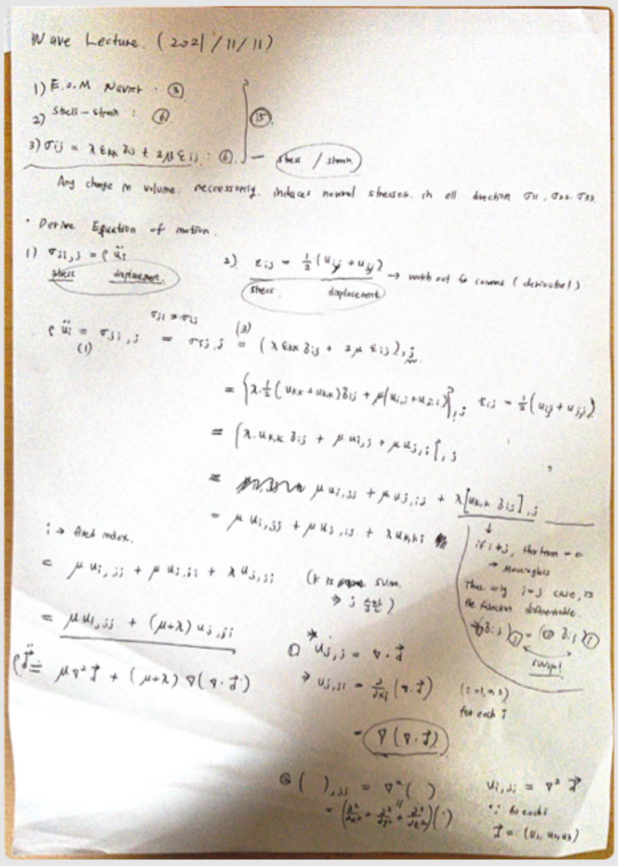
1) ε_{ij} $(i,j = 1,2,3)$ $\quad$ 6 $\quad$ strain

2) σ_{ij} $(i,j = 1,2,3)$ $\quad$ 6 $\quad$ stress

3) u_i $(i = 1,2,3)$ $\quad$ 3 $\quad$ displacement

Wave Lecture 18 (20211111)

스캔된 문서



Wave Lecture (2021/11/11)

1) E.o.M Navier : ③
2) Stress – strain : ⑥
3) $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$: ⑥

⑮ – stress / strain

Any change in volume necessarily induces normal stresses in all direction $\sigma_{11}, \sigma_{22}, \sigma_{33}$

- Derive Equation of motion

1) $\sigma_{ji,j} = \rho \ddot{u}_i$ stress displacement

2) $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ → work out to remove (derivative) stress displacement

$\sigma_{ji} = \sigma_{ij}$

$\rho \ddot{u}_i = \sigma_{ji,j} = \sigma_{ij,j} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij})_{,j}$

$= \{\lambda \cdot \frac{1}{2}(u_{k,k} + u_{k,k})\delta_{ij} + \mu(u_{i,j} + u_{j,i})\}_{,j}$, $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$= \{\lambda u_{k,k} \delta_{ij} + \mu u_{i,j} + \mu u_{j,i}\}_{,j}$

$= \mu u_{i,jj} + \mu u_{j,ij} + \lambda [u_{k,k} \delta_{ij}]_{,j}$

$= \mu u_{i,jj} + \mu u_{j,ij} + \lambda u_{k,ki}$

if $i \neq j$, the term = 0 → Meaningless. Thus only $i = j$ case, is the function differentiable. swap!

i → fixed index.

$= \mu u_{i,jj} + \mu u_{j,ji} + \lambda u_{j,ji}$ (k is dummy sum. ⇒ j 순환)

$= \mu u_{i,jj} + (\mu + \lambda) u_{j,ji}$

$\rho \ddot{\vec{u}} = \mu \nabla^2 \vec{u} + (\mu + \lambda) \nabla(\nabla \cdot \vec{u})$

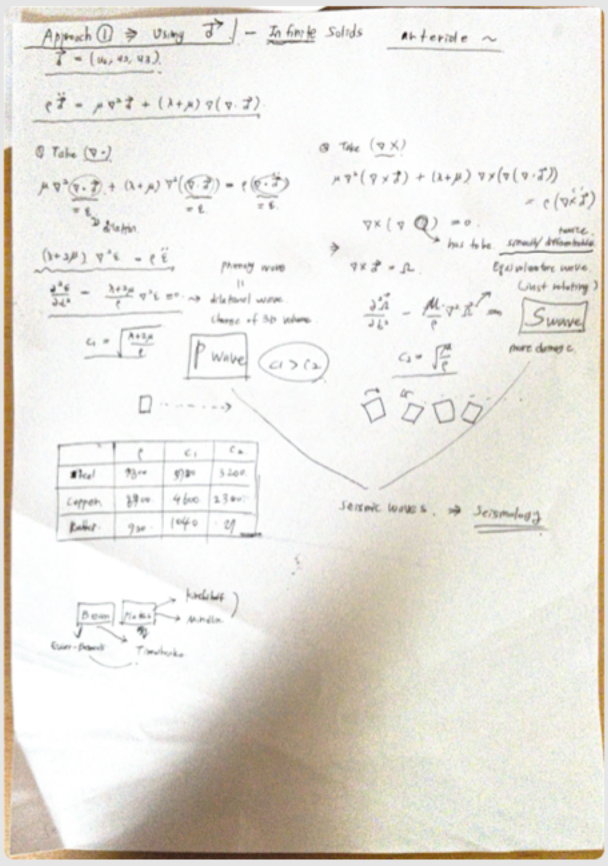
① $u_{j,j} = \nabla \cdot \vec{u}$

⇒ $u_{j,ji} = \frac{\partial}{\partial x_i}(\nabla \cdot \vec{u})$ (i = 1, 2, 3) for each i

$= \nabla(\nabla \cdot \vec{u})$

② $(\ )_{,jj} = \nabla^2(\ ) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)(\ )$

$u_{i,jj} = \nabla^2 \vec{u}$ ∵ for each i $\vec{u} = (u_1, u_2, u_3)$



Approach ① ⇒ Using $\vec{u}$ – Infinite Solids

Arteride ~

$\vec{u} = (u_1, u_2, u_3)$

$\rho \ddot{\vec{u}} = \mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla(\nabla \cdot \vec{u})$

① Take $(\nabla \cdot)$

$\mu \nabla^2 (\nabla \cdot \vec{u}) + (\lambda + \mu) \nabla^2 (\nabla \cdot \vec{u}) = \rho (\nabla \cdot \ddot{\vec{u}})$, $= \varepsilon$ ⇒ dilatation

$(\lambda + 2\mu) \nabla^2 \varepsilon = \rho \ddot{\varepsilon}$

$\frac{\partial^2 \varepsilon}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \nabla^2 \varepsilon = 0$ → dilatational wave

primary wave = dilatational wave, change of 3D volume

$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

P wave

$c_1 > c_2$

② Take $(\nabla \times)$

$\mu \nabla^2 (\nabla \times \vec{u}) + (\lambda + \mu) \nabla \times (\nabla(\nabla \cdot \vec{u})) = \rho (\nabla \times \ddot{\vec{u}})$

$\nabla \times (\nabla Q) = 0$ → has to be smoothly differentiable twice

⇒ $\nabla \times \vec{u} = \Omega$

Equivoluminal wave (just rotating)

$\frac{\partial^2 \vec{\Omega}}{\partial t^2} - \frac{\mu}{\rho} \nabla^2 \vec{\Omega}$

S wave

more damage

$c_2 = \sqrt{\frac{\mu}{\rho}}$

	ρ	c_1	c_2
Steel	7800	5900	3200
Copper	8900	4600	2300
Rubber	930	1040	27

Seismic waves ⇒ Seismology

Beam → Euler-Bernoulli, Timoshenko

Plates → Kirchhoff, Mindlin

① $\sigma_{ji,j} = \rho \ddot{u}_i$ ② $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ ③ $\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu \varepsilon_{ij}$

$$\rho \ddot{u}_i = \sigma_{ij,j} = (\lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij})_{,j} = \left\{\lambda \cdot \tfrac{1}{2}(u_{k,k} + u_{k,k})\delta_{ij} + 2\mu\varepsilon_{ij}\right\}_{,j}$$

$$= \left\{\lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})\right\}_{,j} = \lambda u_{k,ki} + \mu(u_{i,j} + u_{j,i})_{,j}$$

$$(\lambda u_{k,k}\delta_{ij})_{,j} = \lambda u_{k,kj} = \lambda u_{k,ki}$$

($\because \delta_{ij} = 0$ if $i \neq j \Rightarrow$ meaningless. Consider only $i = j$ case $\rightarrow \delta_{ij} = 1$)

$$\Rightarrow \rho \ddot{u}_i = \lambda u_{k,ki} + \mu u_{i,jj} + \mu u_{j,ij} = \lambda u_{j,ji} + \mu u_{i,jj} + \underbrace{\mu u_{j,ij}}_{= \mu u_{j,ji}}$$

$$\Rightarrow \rho \ddot{u}_i = \mu u_{i,jj} + (\mu + \lambda) u_{j,ji}$$

① $u_{j,j} = \nabla \cdot \vec{d} \Rightarrow u_{j,ji} = \nabla(\nabla \cdot \vec{d})$

② $u_{i,jj} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_i = \nabla^2 u_i$

$$\Rightarrow \rho \ddot{\vec{d}} = \mu \nabla^2 \vec{d} + (\mu + \lambda)\nabla(\nabla \cdot \vec{d})$$

(Note that $\nabla^2 f = \nabla \cdot (\nabla f)$ is conventionally used for scalar field f only.)

☆ $\Rightarrow$ Here Laplacian ∇^2 is for each component u_1, u_2, u_3.

① $(\nabla \cdot)$ operator

$$\rho \nabla \cdot \ddot{\vec{d}} = \mu \nabla^2 \underbrace{(\nabla \cdot \vec{d})}_{= w} + (\mu + \lambda)\nabla^2(\nabla \cdot \vec{d})$$

$$\Rightarrow \frac{\partial^2 w}{\partial t^2} - \frac{2\mu + \lambda}{\rho}\nabla^2 w = 0$$

$$C_p = \sqrt{\frac{2\mu + \lambda}{\rho}}$$

② $(\nabla \times)$ operator.

$$\rho(\nabla \times \ddot{\vec{d}}) = \mu \nabla^2 \underbrace{(\nabla \times \vec{d})}_{= \Omega} + (\mu + \lambda)\underbrace{\nabla \times \nabla}_{= 0}(\nabla \cdot \vec{d})$$

$$\Rightarrow \frac{\partial^2 \Omega}{\partial t^2} - \frac{\mu}{\rho}\nabla^2 \Omega = 0$$

$$C_s = \sqrt{\frac{\mu}{\rho}}$$

Bulk modulus

$\nabla \cdot \vec{d}$: compression : P wave

$\nabla \times \vec{d}$: spin/twist : S wave

Seismic waves.

Note : These results are for infinitely long solids.

Wave Lecture 19 (20211116)

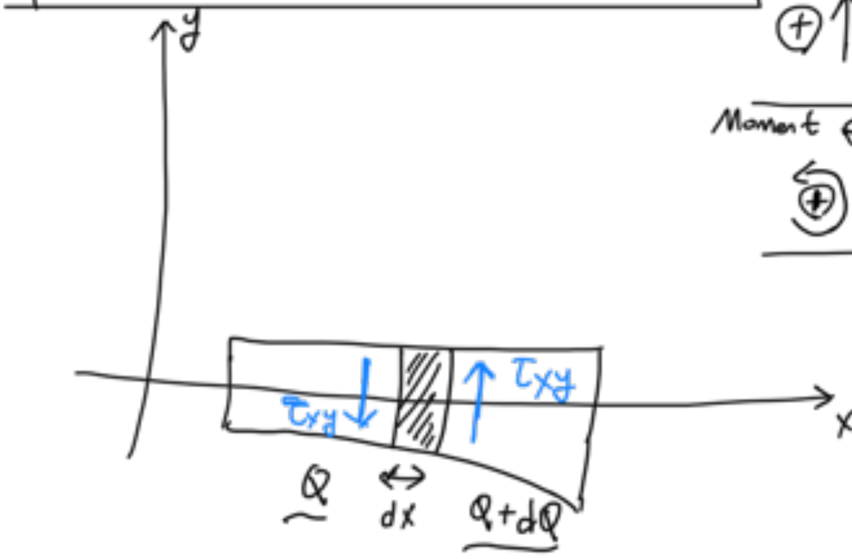
String / membrane | Beams. (Euler - Bernoulli → Timoshenko)

Infinite elastic solids. | Plate (Kirchhoff → Mindlin)

⇒ Approximate theory of elastic wave motions

∵ Elastic wave motions related to "Vibrations" → noise, → heat

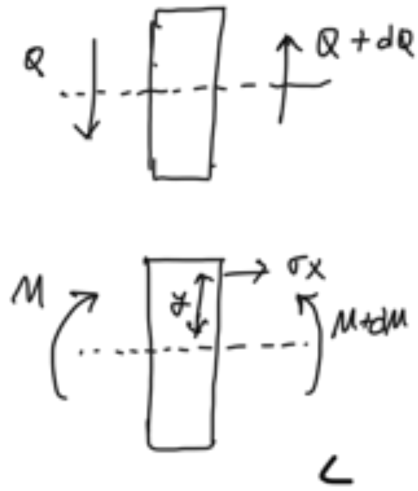
1. Euler - Bernoulli beam theory



$$\oplus\uparrow \quad \underset{\text{force}}{Q} = \int_A \underset{\text{stress}}{\tau_{xy}}\, dA$$

$$\text{Moment} \quad \circlearrowleft\oplus \quad M = \int_A -\sigma_x \cdot y \, dA$$

Effect of τ_{xy} to M will be concerned later.



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* Nomenclature (명명법)

① $w(x,t)$: transverse deflection of an infinitesimal element.

② $\theta(x,t)$: rotation angle / slope of centroidal axis. (ccw : +)

③ $Q(x,t)$: shear force

$M(x,t)$: bending moment.

④ ρ : density / E : Young's modulus.

⑤ $A(x)$: cross-sectional area / $I(x)$ = moment of inertia of area = $\int y^2 dA$.

$J(x)$ = rotational inertia = $\int r^2 dm$.

Assumptions

① Beam has a $\underbrace{\text{straight centroidal axis}}_{(x\text{-axis})}$. and cross-section has $\underbrace{\text{axis of symmetry}}_{(y\text{-axis})}$.

Timoshenko

② Material is elastic / isotropic / homogeneous / continous

$E_x = E_y = E$ = no holes.

③ Transverse deflections are small ($w \ll 1$) $\rightarrow$ Linearization!

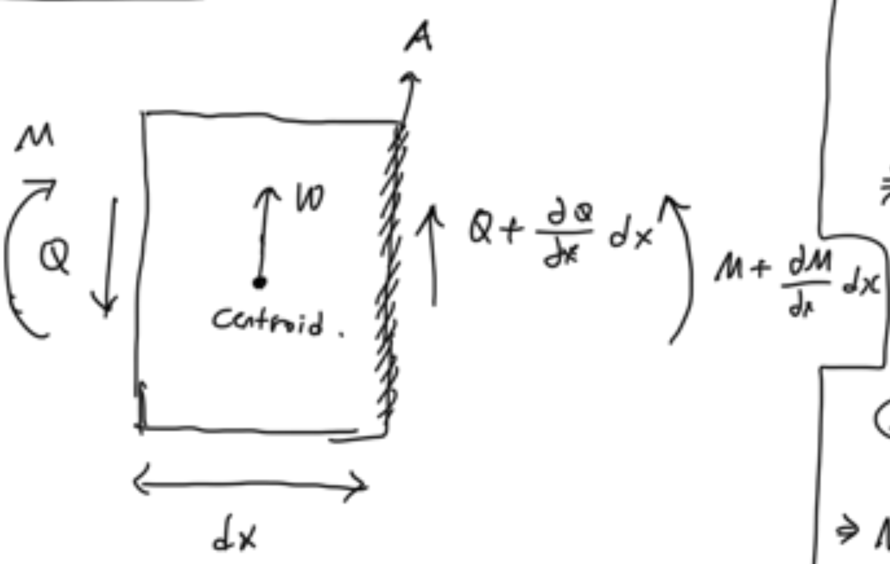
E.B.

④ Transverse deflections are close to purely translational motion in y direction.
(Thus, rotational inertia can be neglected)
$J\ddot{\theta} = 0$

⑤ Cross sections remain plane / perpendicular to x-axis. during/after deformation.
(not distorted) Neglect shear deformation.

Discard ④, ⑤ $\Rightarrow$ Timoshenko Beam

Equations



① $\Sigma F_y = dm \; \partial^2 w / \partial t^2$

$\Rightarrow Q + \frac{\partial Q}{\partial x} dx - Q = \rho A \, dx \; \partial^2 w/\partial t^2$

$\Rightarrow \frac{\partial Q}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2}$

② $\Sigma M = J \, \partial^2 \theta / \partial t^2 = 0$. (for E.B. beams).

$\Rightarrow M + \frac{\partial M}{\partial x} dx - M + \left(Q + \frac{\partial Q}{\partial x} dx\right)\frac{dx}{2} + Q \cdot \frac{dx}{2} = 0$

$\Rightarrow \frac{\partial M}{\partial x} = -Q \quad (\because (dx)^2 \ll 1)$.

Recall that $M = E \cdot I \frac{\partial^2 w}{\partial x^2}$

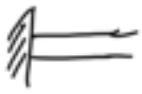
$$\Rightarrow \rho A \frac{\partial^2 w}{\partial t^2} = \frac{\partial Q}{\partial x} = -\frac{\partial^2 M}{\partial x^2} = -\frac{\partial^2}{\partial x^2}\left(E \cdot I \frac{\partial^2 w}{\partial x^2}\right).$$

$$\Rightarrow \rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E \cdot I \cdot \frac{\partial^2 w}{\partial x^2}\right) = 0$$ (Euler-Bernoulli beam equation.)

Boundary conditions. (w, θ, Q, M) at $x = 0$:

① Simply supported. (SS) $w = 0$, $M = 0$

② Free (F)  $Q = 0$, $M = 0$

③ Clamped (C)  $w = 0$, $\theta = 0$ choose. $\binom{4}{2} = 6 \rightarrow 4$ B.C.s / 1 beam.

④ Sliding (S)  $\theta = 0$, $Q = 0$

Wave Lecture 20 (20211118)

Waves in Solids (E−B beam) : $\frac{\partial^2 w}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} = 0$ ✓

Constant for prismatic beam.

For prismatic beam)

B.Cs :
- SS : $w = 0$, $M = 0$ → $w = 0$, $w'' = 0$
- F : $Q = 0$, $M = 0$ → $w''' = 0$, $w'' = 0$
- C : $w = 0$, $\theta = 0$ → $w = 0$, $w' = 0$
- S : $\theta = 0$, $Q = 0$ → $w' = 0$, $w''' = 0$

$M = EI\, w''$

$Q = -\frac{\partial M}{\partial x} = -EI\, w'''$

Note that for acoustic perturbation, 1) $\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0 \Leftrightarrow$ 2) $\frac{\partial^2 w}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} = 0$

1) $p' = A e^{j(kx-\omega t)} \Rightarrow A e^{j(kx-\omega t)}\{(-j\omega)^2 - c^2(jk)^2\} = 0 \Rightarrow \omega = ck \Rightarrow C_p = C$

[Constant in all frequencies (ω)]

2) $w = W e^{j(kx-\omega t)} \Rightarrow (-j\omega)^2 + \frac{EI}{\rho A}(jk)^4 = 0 \Rightarrow \omega = \sqrt{\frac{EI}{\rho A}}\, k^2$

< Dispersion Relation in E.B. beam >

$\Rightarrow C_B = \sqrt[4]{\frac{EI}{\rho A}} \cdot \sqrt{\omega}$ $C_B \sim \sqrt{\omega}$

$\omega \uparrow$: $C_B \uparrow$ →

$\omega \downarrow$: $C_B \downarrow$ →

[Phase speed of elastic waves in E−B is depending on frequency.]

$k^4 = \frac{\rho A}{EI}\omega^2 \Rightarrow k = \pm k_B, \pm j k_B \Rightarrow w(x,t) = \{C_1 e^{jk_B x} + C_2 e^{-jk_B x} + C_3 e^{-k_B x} + C_4 e^{k_B x}\} \cdot (e^{-j\omega t})$

k_B is called "Bending wavenumber"   General solution

* General solution.

$$W(x,t) = \{\underbrace{C_1 e^{jk_B x} + C_2 e^{-jk_B x}}_{\text{Travelling (Propagating) waves}} + \underbrace{C_3 e^{k_B x} + C_4 e^{-k_B x}}_{\text{Evanescent wave}}\} e^{-j\omega t}$$

☆ No exponentially "growing" allowed

- Recall speeds in elastic solids ($C_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$, $C_2 = \sqrt{\frac{\mu}{\rho}}$, $C_B = \sqrt[4]{\frac{EI}{\rho A}}\,\omega$)

Express in E, ν — P wave, S wave

$C_1 = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(2-\nu)}}$ $C_2 = \sqrt{\frac{E}{2\rho(1+\nu)}}$

※ Phase speeds in elastic solids

Navier's

$\Big(c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$, $c_2 = \sqrt{\frac{\mu}{\rho}}$, $\boxed{c_B = \sqrt[4]{\frac{EI}{\rho A}}\sqrt{\omega}} \Big)$

λ, μ : Lamé

$\Downarrow$

E, ν

P-wave

dilatational wave

S-wave (equivol.)

$\boxed{c_1 = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(2-\nu)}}}$, $\boxed{c_2 = \sqrt{\frac{E}{2\rho(1+\nu)}}}$

$\nu = 0$

1D (rod) $\nu = 0$

$\boxed{c_L = \sqrt{\frac{E}{\rho}}}$

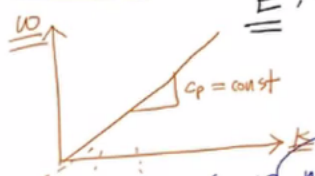
phase speed of longitudinal wave in rod

$\sigma_x = E\,\varepsilon_x$

"transverse" wave

c.f. $\perp$ perpendicular

1D // parallel.



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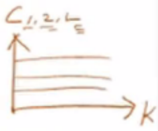
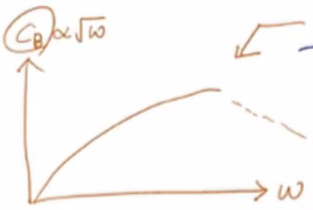
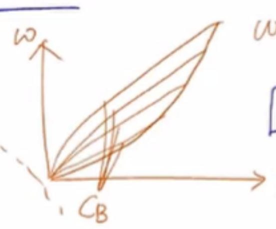


c_1, c_2, c_L $\not\propto \omega$ non-dispersive wave

$c_B \sim \omega$: dispersive wave



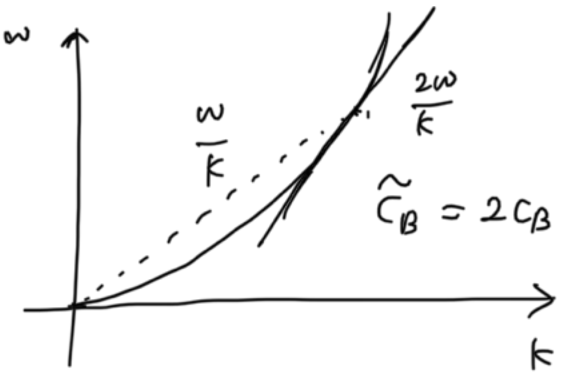


$\omega = \sqrt{\frac{EI}{\rho A}}\, k^2$

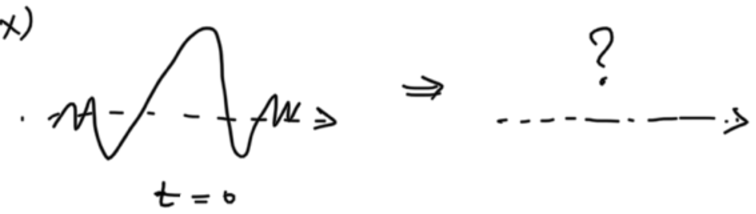
phase

$c_B = \sqrt{\frac{EI}{\rho A}}\, k$

$\tilde{c}_B = \sqrt{\frac{EI}{\rho A}}\, k \cdot 2$



Ex)



$\because \left.\begin{matrix} C_B \sim \omega \\ \tilde{C}_B \sim 2\omega \end{matrix}\right\} \Rightarrow$ $\boxed{\text{wave form not preserved}}$

↓

<u>Dispersive</u>

- <u>Prismatic beam of length "L"</u>

G.E. $\frac{\partial^2 w}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} = 0 \Rightarrow \begin{cases} c_1 e^{jk_B x} + c_2 e^{-jk_B x} + c_3 e^{k_B x} + c_4 e^{-k_B x} \\ b_1 \cos k_B x + b_2 \sin k_B x + b_3 \cosh k_B x + b_4 \sinh k_B x \end{cases}$

B.C. $w(0,t) = 0 \quad w'(0,t) = 0$

$w(L,t) = 0 \quad w'(L,t) = 0$

$$\Rightarrow \left.\begin{aligned} w(0) &= b_1 + b_3 = 0 \\ w'(0) &= k_B b_2 + k_B b_4 = 0 \\ w(L) &= \text{———} = 0 \\ w'(L) &= \text{———} = 0 \end{aligned}\right\}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cos k_B L & \sin k_B L & \cosh k_B L & \sinh k_B L \\ -\sin k_B L & \cos k_B L & \sinh k_B L & \cosh k_B L \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}}_{\vec{b}} = \vec{0}$$

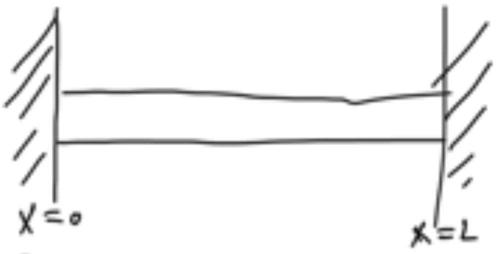
Since $b_1 = b_2 = b_3 = b_4 = 0$ is not considered, <u>$\det(A) = 0$</u>

$$\det(A) = \{(\cosh^2 k_B L - \sinh^2 k_B L) + \sin k_B L \sinh k_B L - \cos k_B L \cosh k_B L\} \cdot 1$$
$$+ \{\cos^2 k_B L + \sin^2 k_B L - \sin k_B L \sinh k_B L - \cos k_B L \cosh k_B L\} \cdot 1 = 0.$$

$$\Rightarrow 2 - 2\cos k_B L \cosh k_B L = 0 \Rightarrow \boxed{\cos k_B L \cosh k_B L = 1}$$ Characteristic Equation

Wave Lecture 21 (20211123)

Waves in E-B beam.



C-C prismatic beam.

clamped.

$\cos k_B L \cosh k_B L = 1$

where $k_B = \sqrt[4]{\frac{\rho A}{EI}}\sqrt{\omega}$

$\Rightarrow k_B L = 4.73 \ldots$

$7.85 \ldots$

$10.99 \ldots$

$14.13 \ldots$

" Temporal " Eigenfrequency

$f_1 = \frac{4.73^2}{2\pi}\sqrt{\frac{EI}{\rho A}}$

$f_2 = \frac{7.85^2}{2\pi}\sqrt{\frac{EI}{\rho A}}$

$f_3 = \frac{10.99^2}{2\pi}\sqrt{\frac{EI}{\rho A}}$

$\vdots$

geometric features
material properties $\Rightarrow$ f

From previous.
Equations of motion. /

$b_1 + b_3 = 0$, $b_2 + b_4 = 0$

$b_1(\cos k_B L - \cosh k_B L) + b_2(\sin k_B L - \sinh k_B L) = 0$

$\Rightarrow b_3 = -b_1 . \; b_4 = -b_2 .$

$b_2 = \underbrace{\frac{\cos k_B L - \cosh k_B L}{\sin k_B L - \sinh k_B L}}_{\sigma \,\to\, \sigma_n} b_1$

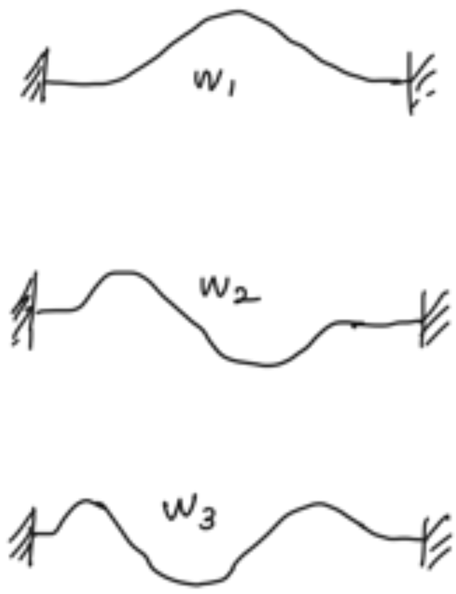
$k_{B,n} = \sqrt[4]{\frac{\rho A}{EI}} \cdot \sqrt{\omega}$

$\Rightarrow W(x,t) = b_1 \{\cos k_B L - \sigma \sin k_B L - \cosh k_B L + \sigma \sinh k_B L\} e^{-j\omega t}$

$b_{1,n}$ $\quad k_{B,n}$ $\quad k_{B,n}$ $\quad k_{B,n}$ $\quad k_{B,n}$

$\Rightarrow W(x,t) = \sum_{n=1}^{\infty} W_n(x,t)$

"Spectral" mode shape



$\vdots$

String (1D) $\neq$ Beam (1D)

HW) compare eigenfrequencies of string & beam for same length

$n=1 \rightarrow f_{beam} =$ $\qquad f_{string} = \frac{1}{2L} \cdot c = \frac{1}{2L}\sqrt{T/\mu}$

Questions! for E-B beams.

No upper bound $\lim_{\omega\to\infty} C_B = \infty$?

· $\boxed{\text{Limitation}}$ of E-B beam theory

From the assumptions of

✓ 4) rotational inertia neglected

✓ 5) shear deformation neglected. $J\ddot{\theta} = 0$

Timoshenko theory

$$\Sigma F_y = \rho A \cancel{dx}\, \partial^2 w/\partial t^2 = \cancel{Q} + \frac{\partial Q}{\partial x}\cancel{dx} - \cancel{Q} \Rightarrow \underline{\partial Q/\partial x = \rho A\, \partial^2 w/\partial t^2} \quad \text{— (1)}$$

$$\Sigma M = \frac{\partial M}{\partial x}dx + Q\,dx = \underbrace{\rho I}_{=J}\, dx\, \frac{\partial^2 \psi}{\partial t^2} \quad \text{(2)}$$

$$M = EI\frac{\partial \psi}{\partial x} \quad \text{— (3)}$$

ψ : total angle / γ_0 = shear angle (= 0 for EB beam)

$$\psi = \frac{\partial w}{\partial x} - \gamma_0 \quad \text{— (4)}$$

$$Q = \int_A \tau_{xy}\, dA = G\int_A \gamma_{xy}\, dA = kGA\gamma_0$$

k → correction factor

- k (rectangular) : $\frac{10(1+\nu)}{12+11\nu}$
- k (circular) : $\frac{6(1+\nu)}{7+6\nu}$

$$\Rightarrow Q = kGA(\partial w/\partial x - \psi) \quad \text{— (5)}$$

(1) $\partial Q/\partial x = \rho A\, \partial^2 w/\partial t^2$

(2) $\partial M/\partial x + Q = \rho I\, \partial^2 \psi/\partial t^2$

(3) $M = EI\, \partial \psi/\partial x$

(4) $\psi = \partial w/\partial x - \gamma_0$

(5) $Q = KGA\,\underbrace{(\partial w/\partial x - \psi)}_{=\gamma_0}$

$$\boxed{\begin{aligned} &(1),(5)\quad \frac{\partial}{\partial x}\left(kG\cancel{A}\left(\partial w/\partial x - \psi\right)\right) = \rho \cancel{A}\, \partial^2 w/\partial t^2 && \text{— ①} \\ &(2),(3),(5)\quad \frac{\partial}{\partial x}\left(EI\,\partial \psi/\partial x\right) + kGA\left(\partial w/\partial x - \psi\right) = \rho I \frac{\partial^2 \psi}{\partial t^2} && \text{— ②} \end{aligned}}$$

"Timoshenko" beam equation (2 vars, 2 eqns)
↓
w, ψ

HW) ① Set $\rho I\, \partial^2\psi/\partial t^2 \to 0$ Derive E.B. beam.
② Set $\gamma_0 \to 0$

Prismatic Timoshenko beam theory (EI = const, A = const.)

① $\partial \psi/\partial x = \partial^2 w/\partial x^2 - \frac{\rho}{kG}\, \partial^2 w/\partial t^2$

② $EI\, \partial^2 \psi/\partial x^2 + kGA\left(\partial w/\partial x - \psi\right) = \rho I \frac{\partial^2 \psi}{\partial t^2}$

$$\Rightarrow EI\, \partial^3\psi/\partial x^3 + kGA\left(\partial^2 w/\partial x^2 - \partial \psi/\partial x\right) = \rho I \frac{\partial}{\partial x}\frac{\partial^2 \psi}{\partial t^2}$$

⇒ ψ 에 ① 대입 → Calculations ...

$$\Rightarrow \underbrace{\frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2}}_{\text{E.B beam}} - \underbrace{\frac{I}{A}}_{\text{Rotational inertia}}\underbrace{\left(1 + \frac{E}{Gk}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho I}{kAG}\frac{\partial^4 w}{\partial t^4}}_{\text{Shear deformation}} = 0$$

Q) Which is more important?

Dispersion relation $\quad w = A e^{i(\gamma x - \omega t)}$

$$\frac{EI}{\rho A}\gamma^4 - \omega^2 - \frac{I}{A}\left(1+\frac{E}{Gk}\right)\gamma^2\omega^2 + \frac{\rho I}{kAG}\omega^4 = 0$$

$c_B \sim \sqrt{\omega}$ (Using $\omega = c\gamma$)

Tedious derivation (math)

$$\Rightarrow \bar{c}^4 - \left(1+\frac{Gk}{E}\right)\bar{c}^2 + \frac{Gk}{E} = 0 \quad \left(\bar{c} = \frac{c}{c_L}, \; c_L = \sqrt{\frac{E}{\rho}}\right)$$

$$\Rightarrow \bar{c} = 1 \text{ or } \bar{c} = \sqrt{\frac{Gk}{E}} \Rightarrow c = c_L = \sqrt{E/\rho} \text{ or } c = \sqrt{\frac{Gk}{E}}\sqrt{\frac{E}{\rho}} = \sqrt{\frac{Gk}{\rho}},$$

$C = \sqrt{\frac{\mu}{\rho}}\,(\sqrt{k})$

phase speed of elastic waves in T-beam

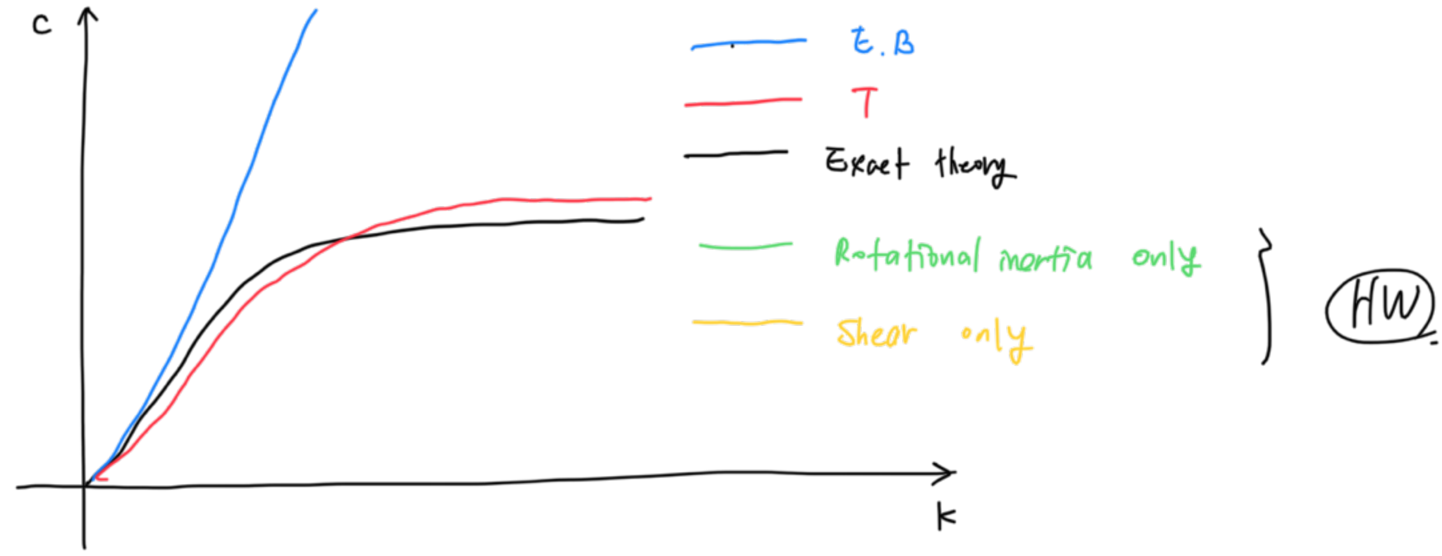
longitudinal

1D rod

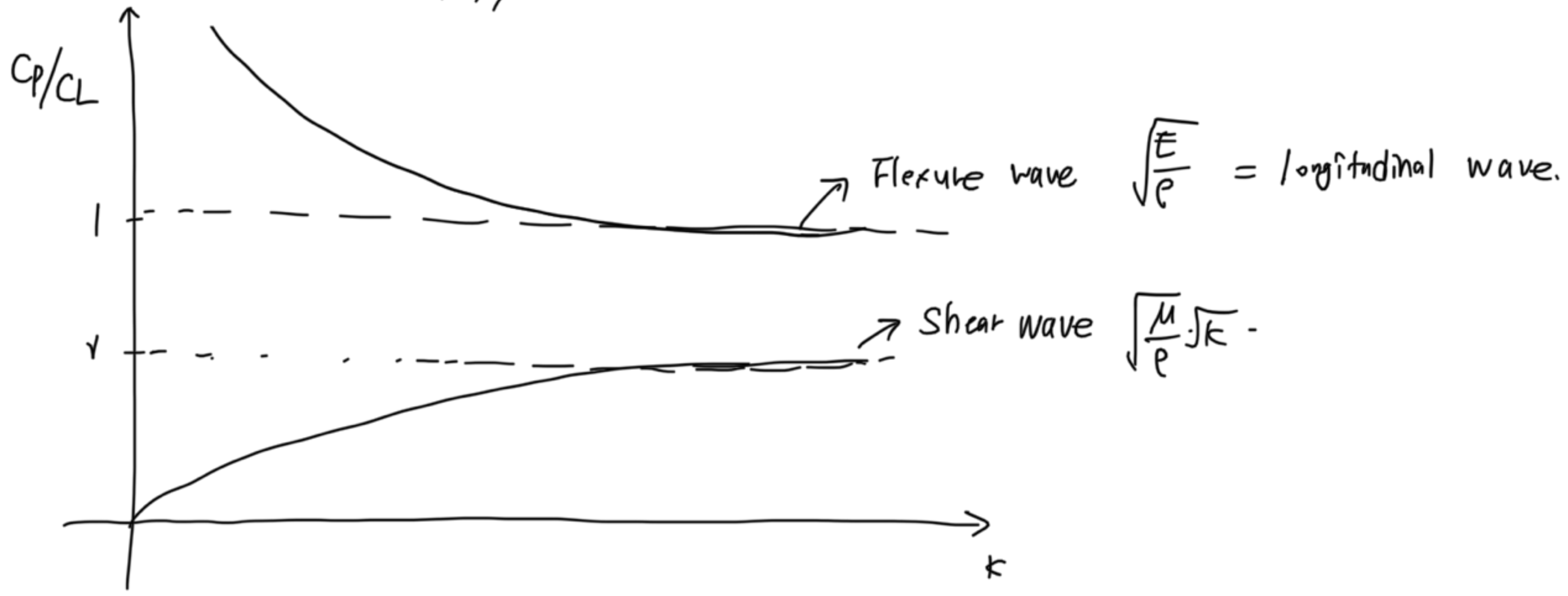
$\sqrt{\frac{\mu}{\rho}}\sqrt{k}$ shear wave

C_2 : S-wave (by shaking)

$\sqrt{\frac{E}{\rho}} \rightarrow$ Flexure wave.



<Timoshenko Beam>



<Rayleigh Beam> : Only considers rotational inertia (not shear strain)

1) $\underline{\rho I \frac{\partial^2 \psi}{\partial t^2}}$ 2) $\underline{\psi = \frac{\partial w}{\partial x} - \gamma_0 = \frac{\partial w}{\partial x}}$

$$① - \frac{\partial}{\partial x}② \Rightarrow \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(\rho I \frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial x}\right) = 0$$

For Prismatic Rayleigh beam. EI, ρA, ρI are constants.

$$\Rightarrow \underline{EI\frac{\partial^4 w}{\partial x^4} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0} \rightarrow w = De^{i(kx-\omega t)}$$

→ Solve!

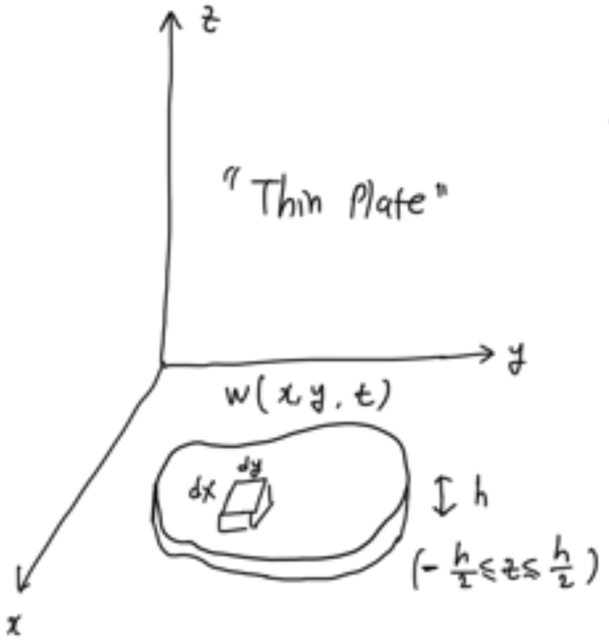
$$\Rightarrow \tilde{\omega}^2 = \frac{\tilde{k}^4}{1+\tilde{k}^2} \quad \left(\text{where} \quad \tilde{\omega} = \frac{\omega\cdot\sqrt{I/A}}{\sqrt{E/\rho}}, \quad \tilde{k} = \sqrt{I/A}\cdot k\right)$$

Wave Lecture 22 (20211125)

Waves in Plates

- Kirchhoff plate
- Mindlin Plate → ① Rotational inertia ② Shear deformation.

Euler B → Timoshenko



- Assumptions

1) Plate has a "flat" mid-plane denoted as "xy plane"

2) Material : Elastic, Isotropic, Homogeneous, and continuous.

3) Transverse deflection $w(x,y,t)$ small $(w \ll 1)$
   ⇒ Linear theory.

Only for Kirchhoff {

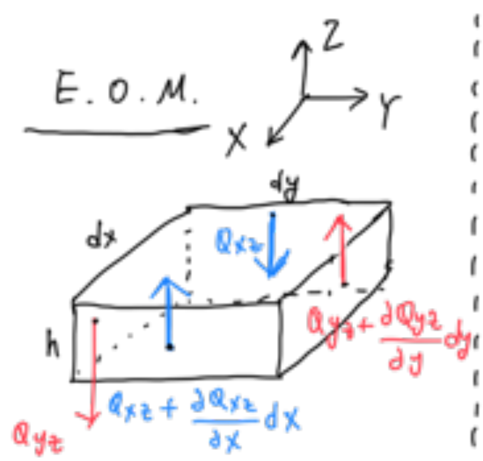
4) Deflection close to a purely translational motion in z-axis (rotational inertia ×)

5) Line elements perpendicular to the mid-plane remain straight & perpendicular to the mid plane during / after deformation.

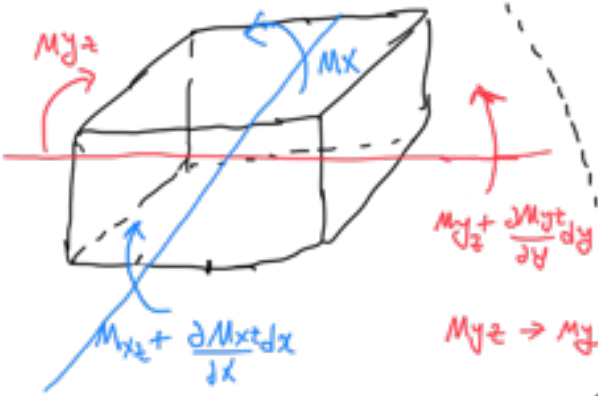
}

6) Line elements do not chage "Length" $(\varepsilon_{zz} = 0)$

E.O.M.

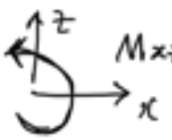


Internal bending moment (per length)	Internal twisting moment (per length.)
1) $M_x \, dy = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_x \, dy \, dz$	3) $M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \, z \, dz$ $\quad dy$
2) $M_y \, dx = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_y \, dx \, dz$	4) $M_{yx} = +\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx} \, z \, dz$ $\quad dx$



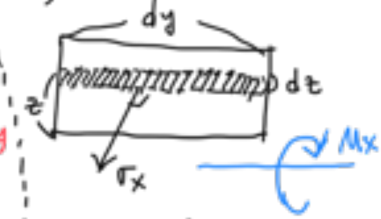
$M_{yz} \to M_y$

$M_{xz} \to M_x$



☆ (Note that M_{xz} (direction of rotation))

1)

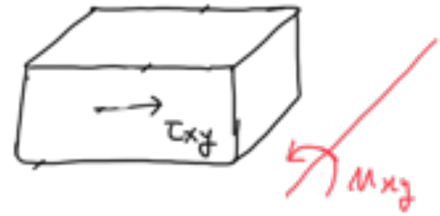


2)

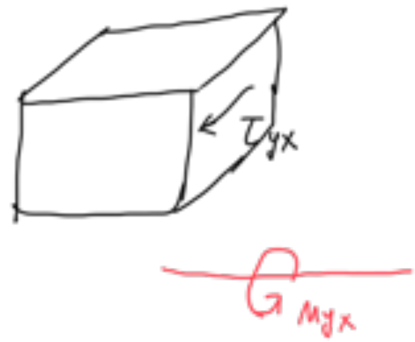
since $\tau_{yx} = \tau_{xy}$

$\Rightarrow M_{xy} = -M_{yx}$

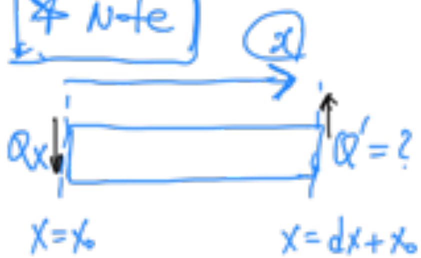
3)



4)



☆ Note



(dx<<1)

$\Rightarrow Q' = Q_x + \frac{\partial Q}{\partial x} \cdot dx$

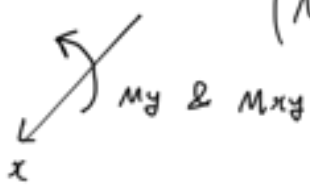
Assuming increase in positive X direction.

① Net force (z)

$$\underbrace{\left(\cancel{Q_x} + \frac{\partial Q_x}{\partial x}dx - \cancel{Q_x}\right)}_{\text{per unit length}}dy + \left(\cancel{Q_y} + \frac{\partial Q_y}{\partial y}dy - \cancel{Q_y}\right)dx = \rho\, dx\, dy\, h\, \partial^2 w/\partial t^2$$

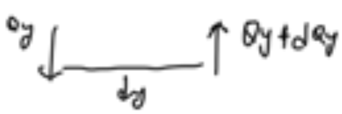
$$\Rightarrow \underline{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h\, \partial^2 w/\partial t^2} \quad -①$$

② Net moment



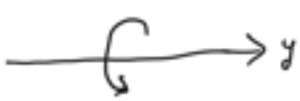
$$\left(M_y + \frac{\partial M_y}{\partial y}dy - M_y\right)dx + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}dx - M_{xy}\right)dy$$

$$+ \left(Q_y + \frac{\partial Q_y}{\partial y}dy + Q_y\right)\frac{dx}{2}dy = J_x = 0.$$



$$\Rightarrow \underline{\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_y = 0} \quad -②$$

③ Net moment



$$-\left(M_x + \frac{\partial M_x}{\partial x} - M_x\right)dy + \left(M_{yx} + \frac{\partial M_{yx}}{\partial y}dy - M_{yx}\right)dx$$

$$-\left(Q_x + \frac{\partial Q_x}{\partial x}dx + Q_x\right)\frac{1}{2}dy \cdot dx = J_y = 0$$

$$\Rightarrow \underline{-\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0} \quad -③$$

③ → ①
② → ① $\Rightarrow$ Q equation

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}\right)=\rho h\,\partial^2 w/\partial t^2$$

$$\Rightarrow -\frac{\partial^2 M_x}{\partial x^2}-\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x \partial y}=\rho h\frac{\partial^2 w}{\partial t^2} \qquad (\because M_{xy}=-M_{yx})$$

Note : Moment - Curvature Constitute Eqns.

Just before, we obtained $-\frac{\partial^2 M_x}{\partial x^2}-\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}=\rho h\frac{\partial^2 w}{\partial t^2}$

※ moment - curvature constitute equations

$M_x = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,\boxed{\sigma_x}\,dz$ where $\boxed{\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \varepsilon_y \nu)}$ under $\boxed{\text{plane stress}}$ condition

$M_y = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,\sigma_y\,dz$ 〃 $\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \varepsilon_x \nu)$

$M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,\tau_{xy}\,dz$ 〃 $\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$

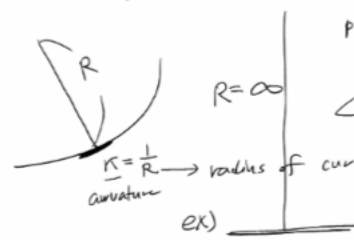
$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

※ stress vector is zero across a particular plane, xy-plane. (in our case)

relations btw $M_{x,y,xy} \sim w \Leftarrow$

here, $\varepsilon_x \cong \underbrace{-z\frac{\partial^2 w}{\partial x^2}}_{\text{deflection}}$ → curvature $\varepsilon_y \cong -z\frac{\partial^2 w}{\partial y^2}$ HW

$\gamma_{xy} \simeq -2z\frac{\partial^2 w}{\partial x \partial y}$



$K = \frac{1}{\infty} = 0$

Then, $M_x = +\int_{-\frac{h}{2}}^{\frac{h}{2}} z\frac{E}{1-\nu^2}\left(+z\frac{\partial^2 w}{\partial x^2}+z\frac{\partial^2 w}{\partial y^2}\nu\right)dz$

$= \frac{E}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\,dz + \frac{\partial^2 w}{\partial y^2}\nu\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\,dz\right]$

$|| \quad 2\int_0^{\frac{h}{2}} z^2\,dz = \frac{2}{3}z^3\Big]_0^{\frac{h}{2}} = \frac{2}{24}h^3 = \frac{h^3}{12}$

$= \frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$ Let $\boxed{D = \frac{Eh^3}{12(1-\nu^2)}}$ be "flexural stiffness"

$M_x = D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$ $M_{xy} = (1-\nu)D\frac{\partial^2 w}{\partial x \partial y}$

$M_y = D\left(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$

$\frac{\partial^2 w}{\partial y^2})$ Let $\boxed{D = \frac{E h^3}{12(1-\nu^2)}}$ be "flexural stiffness"

$xy = (1-\nu) D \frac{\partial^2 w}{\partial x \partial y} \Rightarrow$ EoM: $-\boxed{D}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}\right) w = \rho h \frac{\partial^2 w}{\partial t^2}$

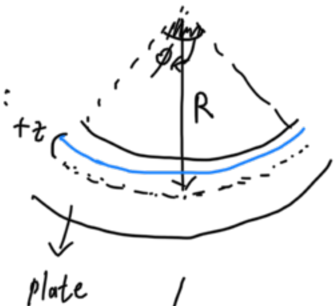
$$\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\nabla^2} \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\nabla^2} = \boxed{\nabla^4}$$

biharmonic operator

Kirchhoff plate equation $\boxed{-D \nabla^4 w = \rho h \frac{\partial^2 w}{\partial t^2}}$

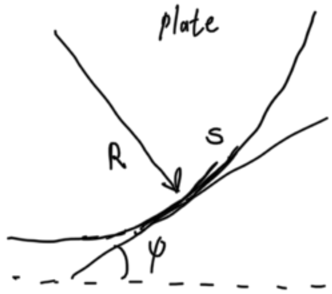
HW : Derive $\begin{cases} \varepsilon_x, \varepsilon_y, \gamma_{xy} \sim w & \text{relation} \\ M_x, M_y, M_{xy} \sim w & \text{relation} \end{cases}$ by yourself.

Note :



$$\varepsilon_x = \frac{\Delta x}{L} = \frac{(R-z)\phi - R\phi}{R\phi} = -\frac{z}{R_x} \qquad R = R_x$$

$$\varepsilon_y = \frac{\Delta y}{L} = \frac{(R-z)\phi - R\phi}{R\phi} = \frac{-z}{R_y} \qquad R = R_y.$$



$$d\psi \cdot R = ds \Rightarrow \frac{1}{R} = \partial\psi/\partial s$$

$$\tan\phi = \partial w/\partial x \Rightarrow \sec^2\psi \frac{\partial\psi}{\partial x} = \partial^2 w/\partial x^2$$

$$ds = \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2}\, dx$$

$$\Rightarrow \partial\psi/\partial s = \partial\psi/\partial x \;\; \partial x/\partial s = \frac{\partial^2 w/\partial x^2}{\sec^2\psi} \cdot \frac{1}{\sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2}}$$

$$\Rightarrow \partial\psi/\partial s = k_x = \frac{1}{R_x} = \frac{\partial^2 w/\partial x^2}{\left\{1 + \left(\frac{\partial w}{\partial x}\right)^2\right\}\sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2}}$$

$$= \frac{\partial^2 w/\partial x^2}{\left[1 + (\partial w/\partial x)^2\right]^{3/2}}$$

Thus, for $(\theta \ll 1 \Leftrightarrow \partial w/\partial x \ll 1)$ case $\Rightarrow \dfrac{1}{R_x} = \dfrac{\partial^2 w}{\partial x^2}$ and $\dfrac{1}{R_y} = \dfrac{\partial^2 w}{\partial y^2}$.

How about γ_{xy} ?

Wave Lecture 23 (20211130)

In Kirchhoff plate, $\underbrace{-D}_{\text{Flexural Stiffness}} \underbrace{\nabla^4 w}_{\nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}} = \rho h \frac{\partial^2 w}{\partial t^2} \Rightarrow \nabla^4 w + \frac{\rho h}{D} \partial^2 w / \partial t^2 = 0$

Let $w(x,y,t) = W e^{j(q_x x + q_y y - \omega t)}$ where $\vec{q} = (q_x, q_y)$: wave vector.

$\Rightarrow q^4 W + \frac{\rho h}{D}(-\omega^2) W = 0 \Rightarrow \omega = \sqrt{\frac{D}{\rho h}}\, q^2 \Rightarrow c_p = \sqrt[4]{\frac{D}{\rho h}}\sqrt{\omega}$

Let $w(x,y,t) = W e^{i(q_x x + q_y y - \omega t)}$. Then, $\nabla^2 w = (iq_x)^2 w + (iq_y)^2 w = -(q_x^2+q_y^2) w = (-q^2) w$

$\nabla^4 w = (-q^2)(-q^2) w = q^4 w$

$\frac{\partial^2 w}{\partial t^2} = (-i\omega)^2 w = -\omega^2 w$

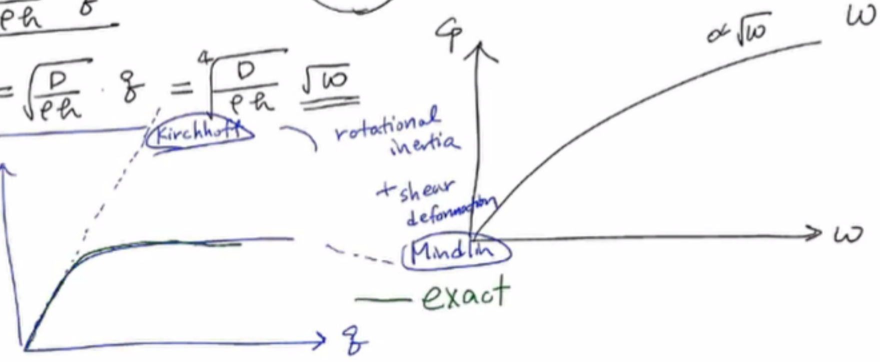
$q^4 W + \frac{\rho h}{D}(-\omega^2) W = 0$

$\Rightarrow q^4 = \frac{\rho h}{D}\omega^2 \Rightarrow \omega = \sqrt{\frac{D}{\rho h}}\, q^2$

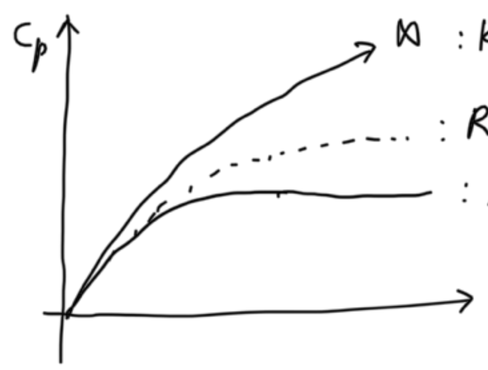
$\Downarrow$

$q = \sqrt[4]{\frac{\rho h}{D}}\sqrt{\omega}$

$c_p = \frac{\omega}{q} = \sqrt{\frac{D}{\rho h}}\cdot q = \sqrt[4]{\frac{D}{\rho h}}\sqrt{\omega}$



w/o bound
$\omega\uparrow, c_p\uparrow^{\infty}$
nonphysical
(c.f. E-B)
limitation of K-plate
$\Downarrow$
Mindlin



: Kirchhoff

: Rayleigh (Considered only rotational inertia)

: Mindlin (Exact)

※ Mindlin plate. (OUT OF SCOPE) (iv) rotational inertia considered (v)

(iv) rotational inertia is taken into account.

(v) Line elements perpendicular to the mid-plane remain straight but not necessarily perpendicular to the mid-plane during and after deformation.

Q : Check Contributions of rotational inertia & shear deformation.

※ Mindlin Plate Theory (w/o derivation)

(iv) rotational inertia is taken into account.

(v) Line elements perpendicular to the mid-plane remain straight but not necessarily perpendicular to the mid-plane during and after deformation.

c.f. Kirchhoff single PDE

EoM (three PDEs - coupled)

$$\begin{cases} \frac{D}{2}\left\{(1-\nu)\nabla^2\psi_x + (1+\nu)\frac{\partial \Phi}{\partial x}\right\} - KGh\left(\psi_x + \frac{\partial w}{\partial x}\right) = \frac{\rho h^3}{12}\frac{\partial^2 \psi_x}{\partial t^2} \\ \frac{D}{2}\left\{(1-\nu)\nabla^2\psi_y + (1+\nu)\frac{\partial \Phi}{\partial y}\right\} - KGh\left(\psi_y + \frac{\partial w}{\partial y}\right) = \frac{\rho h^3}{12}\frac{\partial^2 \psi_y}{\partial t^2} \\ KGh\left(\nabla^2 w + \Phi\right) = \rho h \frac{\partial^2 w}{\partial t^2} \end{cases}$$

rot. accel

— transl. accel.

Q. contributions of
✓ rot. inert
✓ & shear def.

where ψ_x : total angle in x-dir

ψ_y : 〃 in y-dir

$\Phi = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}$

G : shear modulus

K : shear correction factor

$$c_p = \sqrt[4]{Eh^3/12\rho(1-\nu^2)}\,\sqrt{\omega} \quad \cdots$$

W:

- modulus

- correction factor

$k^4 = \frac{\rho h}{D}\omega^2 \rightarrow c_p = \left(\sqrt{\frac{D}{\rho h}}\right)^{\frac{1}{2}}$

$= \sqrt[4]{\frac{D}{\rho h}}\,\sqrt{\omega}$

Plate $\sqrt{\frac{\lambda+2\mu}{\rho}}$ $\sqrt{\frac{\mu}{\rho}}$ $\leftarrow E$

$= \sqrt[4]{\frac{E h^2}{12\rho(1-\nu^2)}}\;\omega\uparrow\;c_p\uparrow$

c_p : phase speed

ρ, E, ν

h

① $E\uparrow$, $c_p\uparrow \propto \sqrt[4]{E}$ — c_1 c_2 $c_L = \sqrt{\frac{E}{\rho}}$

② $\rho\uparrow$, $c_p\downarrow \propto \frac{1}{\sqrt[4]{\rho}}$

③ $h\uparrow\downarrow$: $c_p\uparrow\downarrow \propto \sqrt{h}$

$\sqrt{E}\uparrow$, $c\uparrow$

SAW (Surface Acoustic Waves)

※ Helmholtz decomposition theorem.

Any smooth vector field can be decomposed into the summation of solenoidal & irrotational field
(div - free) (curl - free)

$$\begin{cases} \nabla\cdot F_{solen} = 0 \\ \nabla\times F_{irrot} = 0 \end{cases}$$

vector identities : $\begin{cases} \nabla\times(\nabla\phi) = 0 & \text{scalar fn} \\ \nabla\cdot\nabla\times\psi = 0 & \text{vector fn} \end{cases}$

$F_{irrot} = \nabla\phi$ ⇒ There exist a scalar fn ϕ such that $\nabla\phi = F_{irrot}$ (curl-free)

$F_{solen} = \nabla\times\psi$ 〃 vector-valued fn ψ 〃 $\nabla\times\psi = F_{solen}$

ε P-wave (by pushing) vol. change — dilatational wave

S-wave (by shaking) equivolumetric

$$\Rightarrow \vec{d} = \underbrace{\nabla\phi}_{\text{Dilatational}} + \underbrace{\nabla\times\psi}_{\text{Equivolumetric}}$$

Recall Navier's Eq.

$$\underline{\rho\ddot{\vec{d}} = (\lambda+\mu)\nabla(\nabla\cdot\vec{d}) + \mu\nabla^2\vec{d}}$$

① $\vec{d} = \nabla\phi$ (curl-free, dilatational)

$$\rho\nabla\ddot{\phi} = (\lambda+\mu)\nabla(\nabla^2\phi) + \mu\nabla\nabla^2\phi$$

$$\Rightarrow \rho\ddot{\phi} = (\lambda+2\mu)\nabla^2\phi$$: P-wave Eq.

② $\vec{d} = \nabla\times\psi$ (div-free, equivolume)

$$\rho\nabla\times\ddot{\psi} = (\lambda+\mu)\nabla(\cancel{\nabla\cdot\nabla\times\psi})^{0} + \mu\nabla^2\nabla\times\psi$$

$$\Rightarrow \rho\ddot{\psi} = \mu\nabla^2\psi$$: S-wave Eq.

$\boxed{\text{B.C.s}}$

① No traction force

$\sigma_{yy} = \sigma_{yx} = 0$

② No variation along z axis

(in-plane motion)

xy plane $\rightarrow \frac{\partial}{\partial z} = 0$

$$\vec{d} = \nabla\phi + \nabla\times\psi$$

$$= \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \cancel{\partial\phi/\partial z}^{\,0} \end{pmatrix} + \begin{pmatrix} \frac{\partial\psi_z}{\partial y} - \cancel{\frac{\partial\psi_y}{\partial z}}^{\,0} \\ \cancel{\frac{\partial\psi_x}{\partial z}}^{\,0} - \frac{\partial\psi_z}{\partial x} \\ \frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial\phi}{\partial x} + \frac{\partial\psi_z}{\partial y} \\ \frac{\partial\phi}{\partial y} - \frac{\partial\psi_z}{\partial x} \\ \frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y} \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

{ Only ϕ, ψ_z are needed }

From ①, ②, respectively.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \quad / \quad \frac{\partial^2\psi_z}{\partial x^2} + \frac{\partial^2\psi_z}{\partial y^2} = \frac{1}{c_2^2}\frac{\partial^2\psi_z}{\partial t^2}$$

Wave Lecture 24 (20211207)

- SAW

$$\rho \ddot{\vec{d}} = (\lambda+\mu)\nabla(\nabla\cdot\vec{d}) + \mu\nabla^2\vec{d} \xrightarrow{\text{H.D.T}} \begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = \dfrac{1}{c_1^2}\dfrac{\partial^2\phi}{\partial t^2} & ① \\[2ex] \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = \dfrac{1}{c_2^2}\dfrac{\partial^2\psi}{\partial t^2} & ② \end{cases}$$

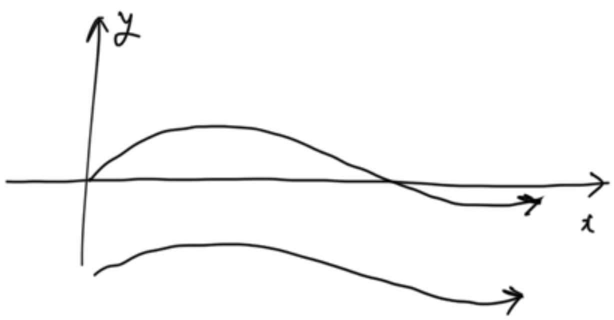
$$\vec{d} = \nabla\phi + \nabla\times\psi$$

B.C.s

1) $\sigma_{yy} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}$ ③

2) $\tau_{yx} = G\gamma_{yx} = 2\mu\varepsilon_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ ④

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi_z}{\partial y}$$
$$v = \frac{\partial\phi}{\partial y} - \frac{\partial\psi_z}{\partial x}$$



Assume $\phi(x,y,t) = f(y)\, e^{j(kx-\omega t)}$
$\psi_z(x,y,t) = g(y)\, e^{j(kx-\omega t)}$

① $\Rightarrow -k^2 f + f'' = -\frac{\omega^2}{c_1^2} f \Rightarrow f'' - \underbrace{\left(k^2 - \left(\frac{\omega}{c_1}\right)^2\right)}_{=\xi^2} f = 0$ $\xi^2 > 0$ (∵ if negative, $f = F(\cos, \sin)$, $y \to -\infty \Rightarrow f = 0$ not possible)

② $\Rightarrow g'' - \underbrace{\left(k^2 - \left(\frac{\omega}{c_2}\right)^2\right)}_{=\eta^2} g = 0$

$f(y) = \cancel{A}e^{-\xi y} + \underline{Be^{\xi y}}$ ($A \to 0$)

$g(y) = \cancel{C}e^{-\eta y} + \underline{De^{\eta y}}$

$$\Rightarrow \boxed{\begin{aligned} \phi &= Be^{\xi y} e^{j(kx-\omega t)} \\ \psi_z &= De^{\eta y} e^{j(kx-\omega t)} \end{aligned}}$$

Note

$$\xi^2 = k^2 - \frac{\omega^2}{c_1^2} = k^2 - \frac{\rho\omega^2}{\lambda+2\mu}$$
$$\eta^2 = k^2 - \frac{\omega^2}{c_2^2} = k^2 - \frac{\rho\omega^2}{\mu}$$

$$\Rightarrow (\lambda+2\mu)\xi^2 = (\lambda+2\mu)k^2 - \rho\omega^2$$
$$\mu\eta^2 = \mu k^2 - \rho\omega^2$$

$$\Rightarrow (\lambda+2\mu)\xi^2 - \lambda k^2 = \mu\eta^2 + \mu k^2 \quad \text{—— } (*)$$

③ : $(\lambda+2\mu)(\xi^2-k^2)B - 2\mu(-k^2B + jkD\eta) = 0$

$$\Rightarrow \left[(\lambda+2\mu)\xi^2 - \lambda k^2\right]B - 2jkD\mu\eta = 0 \underset{\because (*)}{\Rightarrow} \mu(\eta^2+k^2)B - 2jk\mu\eta D = 0$$

④ $\mu(2Bjk\xi + \eta^2 D + k^2 D) = 0$

From ③,④ $\Rightarrow \underbrace{\begin{pmatrix} \eta^2+k^2 & -2jk\eta \\ 2jk\xi & \eta^2+k^2 \end{pmatrix}}_{A} \begin{pmatrix} B \\ D \end{pmatrix} = 0 \Rightarrow \det(A) = 0$

$$\Rightarrow \boxed{(\eta^2+k^2)^2 - 4k^2\xi\eta = 0}$$

Note that, $\eta^2 = k^2 - \dfrac{\omega^2}{C_2^2} = k^2\left(1 - \dfrac{C_R^2}{C_2^2}\right)$ where $C_R = \dfrac{\omega}{k}$ (phase speed of SAW)

$\xi^2 = k^2 - \dfrac{\omega^2}{C_1^2} = k^2\left(1 - \dfrac{C_R^2}{C_1^2}\right)$

$$\Rightarrow \left\{k^2\left(1 - \frac{C_R^2}{C_2^2}\right) + k^2\right\}^2 - 4k^2 \cdot k^2\sqrt{1 - \frac{C_R^2}{C_1^2}}\sqrt{1 - \frac{C_R^2}{C_2^2}} = 0$$

$$\Rightarrow \boxed{\left(2 - \frac{C_R^2}{C_2^2}\right)^4 = 16\left(1 - \frac{C_R^2}{C_1^2}\right)\left(1 - \frac{C_R^2}{C_2^2}\right)}$$ ✓

C_R is independent of frequency (ω)

$\because C_R = C_R(C_1, C_2)$

• Particle Trajectories

Also, $D = \dfrac{-2jk\xi}{\eta^2+k^2} \cdot B$

Recall, $\left.\begin{array}{l} \phi = Be^{\xi y} \cdot e^{j(kx-\omega t)} \\ \psi_z = De^{\eta y} \cdot e^{j(kx-\omega t)} \end{array}\right\}$

$$\boxed{\psi_z = -\frac{2ik\xi}{\eta^2+k^2} e^{\eta y} e^{i(kx-\omega t)}}$$

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi_z}{\partial y} = B\,ik\underbrace{\left[e^{\xi y} - \frac{2\xi\eta}{\eta^2+k^2} e^{\eta y}\right]}_{=a} e^{i(kx-\omega t)}$$

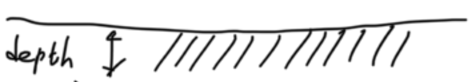
$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi_z}{\partial x} = B\xi\underbrace{\left[e^{\xi y} - \frac{2k^2}{\eta^2+k^2} e^{\eta y}\right]}_{=b} e^{i(kx-\omega t)}$$

$$\Rightarrow \left|\frac{u}{a}\right|^2 + \left|\frac{v}{b}\right|^2 = 2. \quad \rightarrow \text{Ellipse.}$$

① oblate ② prolate

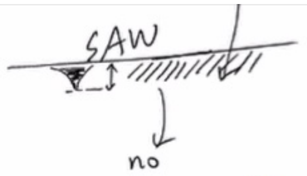
① CW ② CCW

$$\boxed{\text{Penetration Depth}}$$



$$\frac{u(y)}{u(0)} = \frac{e^{\xi y} - \frac{2\xi\eta}{\eta^2+k^2} e^{\eta y}}{1 - \frac{2\xi\eta}{\eta^2+k^2}} \ll 1$$

$$\frac{v(y)}{v(0)} = \frac{e^{\xi y} - \frac{2k^2}{\eta^2+k^2} e^{\eta y}}{1 - \frac{2k^2}{\eta^2+k^2}} \ll 1$$



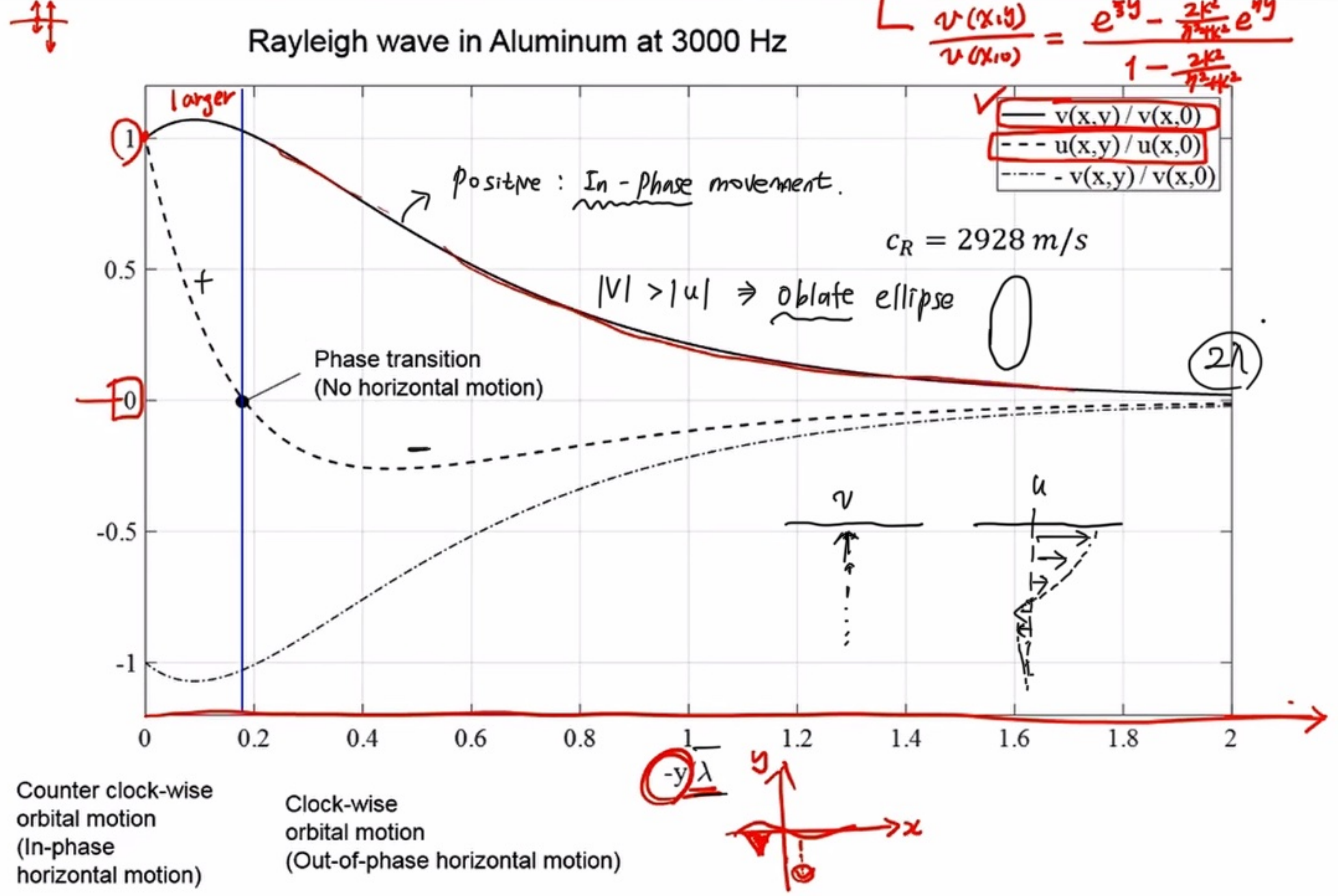
Shap

HW.

$$\left[\begin{aligned} \frac{u(y)}{u(0)} &= \frac{e^{\xi y} - \frac{2\xi\eta}{\eta^2+k^2} e^{\eta y}}{1 - \frac{2\xi\eta}{\eta^2+k^2}} \\ \frac{v(y)}{v(0)} &= \frac{e^{\xi y} - \frac{2k^2}{\eta^2+k^2} e^{\eta y}}{1 - \frac{2k^2}{\eta^2+k^2}} \end{aligned}\right.$$

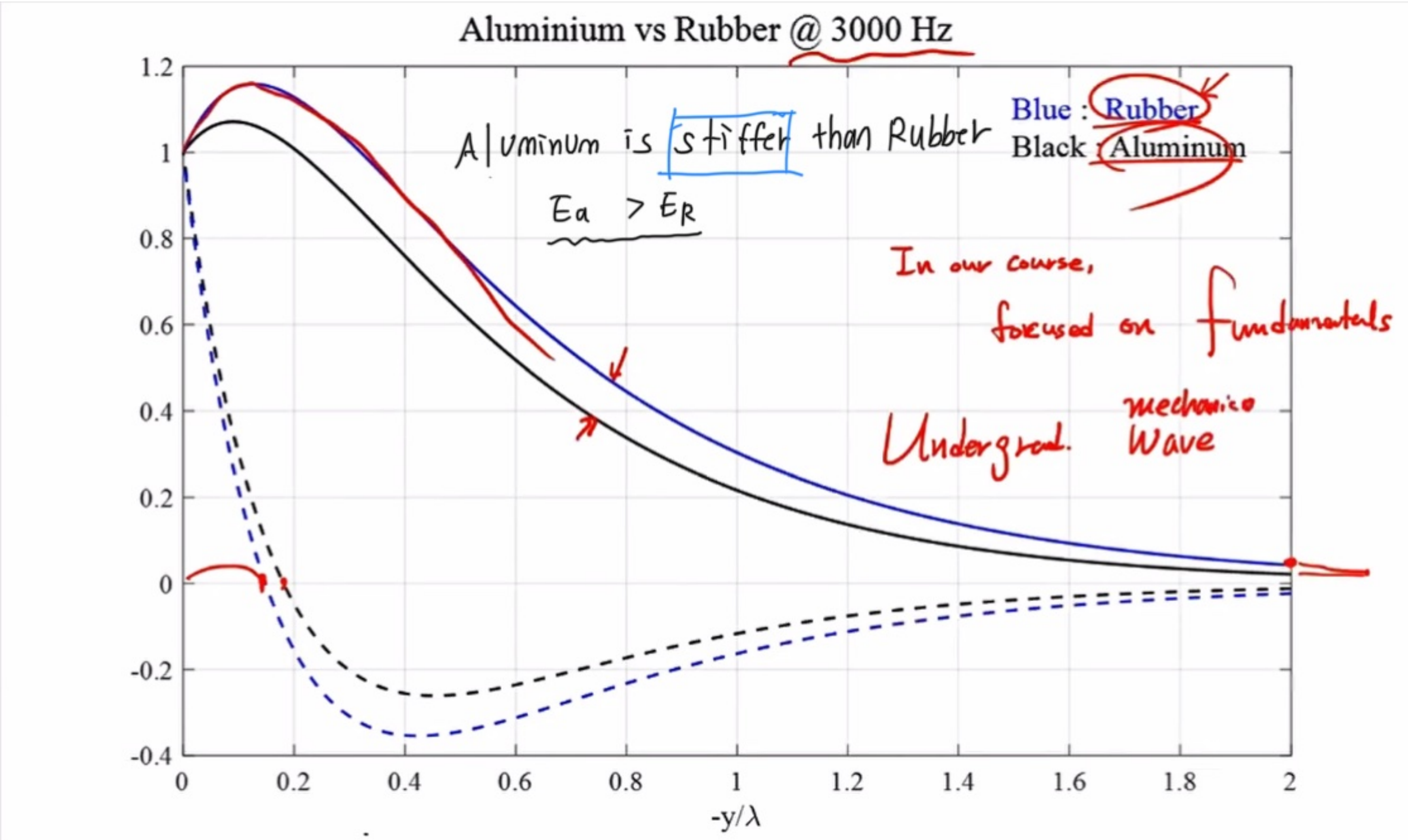
Penetration Depth of Rayleigh Wave

$$\left[\begin{aligned} \frac{u(x,y)}{u(x,0)} &= \frac{e^{\xi y} - \frac{2\xi\eta}{\eta^2+k^2} e^{\eta y}}{1 - \frac{2\xi\eta}{\eta^2+k^2}} \\ \frac{v(x,y)}{v(x,0)} &= \frac{e^{\xi y} - \frac{2k^2}{\eta^2+k^2} e^{\eta y}}{1 - \frac{2k^2}{\eta^2+k^2}} \end{aligned}\right.$$



Counter clock-wise orbital motion (In-phase horizontal motion)

Clock-wise orbital motion (Out-of-phase horizontal motion)



Aluminium vs Rubber @ 3000 Hz
Blue : Rubber
Black : Aluminum
Aluminum is stiffer than Rubber
Ea > ER
In our course,
focused on fundamentals
Undergrad.
mechanics
Wave
-y/λ
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
-0.4
-0.2